Planar Strips and an End Concept for Digraphs

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Zusammenfassung

Diese Arbeit besteht aus zwei Teilen. Im ersten Teil wird ein Endenkonzept für gerichtete Graphen eingeführt und untersucht. Sind Θ und Θ' zwei gerichtete Wege in einem Digraphen Δ, so bedeute a) $\Theta \leq \Theta'$, daß es unendlich viele paarweise disjunkte $\Theta \to \Theta'$-Wege gibt und b) $\Theta \sim \Theta'$, daß sowohl $\Theta \leq \Theta'$ als auch $\Theta' \leq \Theta$ gilt. Da $\leq$ eine Quasiordnung ist, ist $\sim$ eine Äquivalenzrelation, und $\leq$ induziert eine partielle Ordnung auf der Menge der $\sim$-Klassen, die *Enden* genannt werden. Es werden eine notwendige und einige hinreichende Bedingung(en) dafür vorgestellt, daß eine abstrakte Ordnung durch die Endenordnung eines Digraphen darstellbar ist. Des weiteren enthält diese Arbeit Resultate über endengleiche Ober- und Unterdigraphen, Translationen von Digraphen sowie Enden von eckentransitiven Digraphen. Die Hauptergebnisse des ersten Teils sind:

1) Jeder abzählbare Digraph mit einer Anfangsecke, in dem es kein Paar $\leq$-vergleichbarer ω-Enden gibt, enthält eine ω-endengleiche, aufspannende Arboreszenz.

2) Sei Δ ein unendlicher, zusammenhängender und eckentransitiver Digraph. Wenn der Δ zugrundeliegende ungerichtete Graph genau zwei Enden hat, dann ist Δ ein gerichteter Strip. Jeder gerichtete Strip hat genau zwei zusammengesetzte Enden. Ist Δ außerdem lokalfinit oder stark zusammenhängend und hat der Δ zugrundeliegende ungerichtete Graph mindestens $2^{\aleph_0}$ Enden, so hat Δ wenigstens $2^{\aleph_0}$ zusammengesetzte Enden.

Der zweite Teil dieser Arbeit enthält eine vollständige Klassifizierung aller eckentransitiven planaren Strips. Jeder solche Strip hat einen der Eckentypen (3^6), $(3^3.4^2)$, $(3^2.4.3.4)$, (4^4), (6^3), oder (4.8^2). (Der Eckentyp eines ebenen Graphen beschreibt die zyklische Anordnung der Länder um eines Ecke des Graphen, wobei diese Anordnung bei allen Ecken die gleiche sein muß.) Der zweite Parameter der Klassifizierung, κ_∞ (die minimale Anzahl von Ecken, die man aus dem Graphen entfernen muß, um wenigstens zwei unendliche Komponenten zu erhalten), durchläuft $\mathbb{N}$. Bei einigen Eckentypen ist ein dritter Parameter nötig. Dieser Parameter nimmt für festes κ_∞ aber nur endlich viele Werte an.

Abstract

This thesis is divided into two parts. In the first part, an end concept for directed graphs is introduced and examined. For two one-way infinite directed paths Θ and Θ' of a digraph Δ define: a) $\Theta \leq \Theta'$ if there are infinitely many pairwise disjoint directed paths from Θ to Θ'; b) $\Theta \sim \Theta'$ if $\Theta \leq \Theta'$ and $\Theta' \leq \Theta$. The relation $\leq$ is a quasiorder, and hence $\sim$ is an equivalence relation (whose classes are called *ends*). Furthermore, $\leq$ induces a partial order on the set of ends of a digraph. A necessary and some sufficient condition(s) are presented for an abstract order to be representable by the end order of a digraph. Furthermore, there are several results concerning endfaithful super- and subdigraphs, translations of digraphs and ends of vertex-transitive digraphs. The main results of the first part are:

1) Every countable digraph with an initial vertex and no pair of $\leq$-comparable ω-ends contains an ω-endfaithful, spanning arborescence.

2) Let Δ be an infinite, connected, vertex-transitive digraph. If the underlying undirected graph of Δ has exactly two ends, then Δ is a directed strip. Every directed strip has exactly two composed ends. If Δ is locally finite or strongly connected and the underlying undirected graph of Δ has at least $2^{\aleph_0}$ ends, then Δ has at least $2^{\aleph_0}$ composed ends.

The second part of this thesis gives a complete classification of the vertex-transitive, planar strips. It turns out that any such strip has one of the vertex types (3^6), $(3^3.4^2)$, $(3^2.4.3.4)$, (4^4), (6^3), and (4.8^2). (The vertex type of a plane graph describes the cyclic ordering of the faces around a vertex of the graph, where this order has to be the same at each vertex.) The second parameter of the classification, κ_∞ (which is the minimum number of vertices which one has to remove from the graph to obtain at least two infinite components), ranges over $\mathbb{N}$. For some of the vertex-types, a third parameter is necessary. For fixed κ_∞, this parameter ranges over a finite set of values.

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Introduction

In the early 40's, FREUDENTHAL [21], [22] and HOPF [40] investigated discrete groups with the aid of the end concept which, beforehand, had been used to obtain certain closures and compactifications of continuous structures (see, e.g., FREUDENTHAL [20]). Until recently, there occasionally appeared papers referring to the work of HOPF and FREUDENTHAL, for example [75], [76], [10], [11], [39], and [90].

In the mid-sixties, HALIN started a broad attack on the infinite graphs by studying their ray structure (a *ray* is a one-way infinite path) in a series of papers (see [29], [27], [31], [30], [32], [33], [34], [35], [36]). In [29], he independently reintroduced an end concept for graphs based on rays[2] which turned out to be a basic and important tool in infinite graph theory. Despite the fact that HALIN's research left open a lot of interesting problems it only caused a very weak echo till the end of the 70's[3] when POLAT ([61], [62], [64], [63], [65], [66]; see [67] for an overview) started a profound inquiry of the infinite graphs by using a more topological approach.

Among the questions HALIN posed the following is perhaps the one which stirred up the most attention: In his already mentioned fundamental paper [29], HALIN showed that every connected, countable graph Γ has an *endfaithful spanning tree* (i.e. a spanning tree of Γ whose end structure coincides with the end structure of Γ), raising the problem whether all graphs contain such a spanning tree. This problem was solved in the negative sense not until 1989 by SEYMOUR and THOMAS [74] and in 1990 independently by THOMASSEN [79] by constructing counterexamples. Other papers dealing with endfaithful spanning trees are an early work of JUNG [43], the above mentioned paper of POLAT [64] who, in addition, did a lot of further work on this problem (see e.g. [69], [70], and [72]), several inquiries ([12], [14]) and an overview [13] of DIESTEL, some works of ŠIRÁŇ and others ([84], [26], and [83]), and some other articles, for example by SABIDUSSI [73], VESTERGAARD [82], and BROCHET and DIESTEL [9].

The study of the relationships between the ends and the automorphisms of a graph began with the seminal paper [34] of HALIN which contained an abundance in both important

[2] An *end* of a graph Γ is an equivalence class with respect to the following relation on the set of all rays of Γ: Two rays Π, Π' of Γ are called equivalent if they cannot be separated by a finite subgraph of Γ. Note that an end of Γ can be regarded as an ideal vertex of the graph Γ and that it is possible to extend MENGER's Theorem such that it fits also for these ideal vertices (see ZELINKA [92], and [35], [61], [68], [71]). Note further that the ends of a CAYLEY graph of an (infinite) finitely generated group coincide with the ends of the group.

[3] The following problem attracted some attention from the mid-seventies on: Call a graph Γ *normal* if a graph Γ' contains infinitely many pairwise disjoint subdivisions of Γ whenever Γ' contains n pairwise disjoint subdivisions of Γ for every $n \in \mathbb{N}$. In [32] and [33], HALIN showed that rays and 2-rays (two-way infinite paths) are normal. In [36] he proved that every tree with maximum degree 3 is normal. Later, ANDREAE demonstrated in a series of papers ([2], [3], [4], [5], [6]) among other things that also every countable, rayless tree and every locally finite tree is normal. Some other related papers are [53], [91], [81], and [19].

and basic results. Above all, it was the notion of the later so-called *translations* ("automorphisms of type 2" in HALIN's paper) which proved to be powerful in the investigation of infinite graphs, in particular in connection with their end structure. HALIN also derived theorems of the fixed point type as well as results about locally finite graphs with an uncountable automorphism group. Subsequent papers were busy in generalizing and simplifying HALIN's results and proofs (e.g. [45], [16]), in the application of HALIN's concepts to vertex-transitive graphs (e.g. JUNG [44]; JUNG and WATKINS [50]; DIESTEL, JUNG, and MÖLLER [16]), and in obtaining new theorems of the fixed point type (e.g. JUNG [47]). Some of them used methods HALIN did not use (e.g. the fragment concept in [44] and [50]).

The notion of a translation is closely related to a very well known class of 2-ended graphs called *strips* (see e.g. [50]). WATKINS [88] classified all edge-transitive, planar strips. In the second part of this thesis, a classification of all vertex-transitive, planar strips is given. The property of being vertex-transitive is interesting in combination with the end structure of a graph, since every infinite, connected, vertex-transitive graph Γ has either 1, 2, or at least $2^{\aleph_0}$ ends (see [16]). In fact, in the 2-ended case Γ is a strip. In the infinitely ended case, Γ has a sort of tree-like structure which is fairly good to handle, because an essential portion of this tree-like structure is due to translations. MÖLLER [55], [56], [57] studies the infinitely ended case using the concept of structure trees. A lot of interesting results concerning infinite, connected, vertex-transitive, locally finite graphs (not only concerning the infinitely ended case) can be found in [80].

The end structure of a graph provides least information if the graph has only one end. Hence the one-ended case in the above mentioned result is the most difficult one. Consequently, some recent papers and works of JUNG and NIEMEYER (see [46], [48], [58], [59]) study certain refinements of HALIN's end concept. The most promising one is an equivalence relation on the set of rays of a graph which is based on the HAUSDORFF-distance.

In the literature so far, the end notion seems to have been used only in the investigation of *undirected* graphs. The question occurs, whether it is possible to define an analogue of the end notion for digraphs. It is the purpose of the first part of this thesis (Chapter 2) to show that an analogue of "undirected ends" for digraphs makes sense and to point out and to examine some differences to ends in undirected graphs. The end concept for digraphs introduced in Section 2.1 can be regarded both as a generalization of the end notion for undirected graphs as well as a refinement of certain subends of the underlying undirected graph. (Any graph Γ can be understood as a symmetric digraph by substituting each edge of Γ by a directed cycle of length 2, and any digraph Δ can be regarded as a special orientation of the underlying undirected graph.) The following is an important question though it is not answered in this thesis:

Problem 1.
Can "directed ends" be productively used in the investigation of undirected infinite graphs?
$\square$

To state things in a more detailed fashion, note that, for the purpose of distinction, directed paths (resp. cycles) will be called *tracks* (resp. *circuits*). (A comprehensive, detailed, and (hopefully) clearly arranged presentation of the terminology which is used throughout this thesis can be found in Chapter 1.) For two one-way infinite tracks — briefly called *1-tracks* — Θ and Ψ in a digraph Δ let $\Theta \leq_\Delta \Psi$ mean that there are infinitely many pairwise disjoint $\Theta \to \Psi$–tracks and $\Theta \sim_\Delta \Psi$ mean that $\Theta \leq \Psi$ and $\Psi \leq \Theta$. Then $\leq$ is a quasiorder and $\sim$ is an equivalence relation on the set of all 1-tracks of Δ. The classes with respect to $\sim$

are also called *ends*, and $\leq$ establishes a partial order on the set of ends of Δ. In Section 2.1 some basic results on ends in digraphs are proved and some differences to ends in undirected graphs are pointed out. For example, if 1-tracks are comparable, they cannot be separated by a finite subdigraph, but nevertheless they can belong to different ends. (In an undirected graph, however, every pair of distinct ends can be separated by a finite subgraph.) Moreover, there are three different end-notions for digraphs: ω-ends (containing only 1-tracks *going* to infinity), ω^*-ends (containing only 1-tracks *coming* from infinity), and composed ends (possibly containing both types of one-way infinite tracks). Furthermore, if $\mathcal{E}$ is an end and S is a strong component of the digraph Δ, then either a rest of every 1-track $\in \mathcal{E}$ is contained in S, or every 1-track $\in \mathcal{E}$ has only finitely many vertices in common with S.

Under these circumstances, one may ask whether it is justified to call the $\sim$-classes "ends", since they are not ends in the topological sense. Nevertheless, these "directed ends" are a generalization of the "undirected" end notion, and they reflect the ramification structure of the 1-tracks in a natural way. However, for a full justification of this naming, a sort of "directed topology" ought to be developed.

In Section 2.3, the following question — which is not very interesting for undirected graphs — is considered: Which types of abstract orders can arise on sets of ends of digraphs? An abstract poset $(X,\leq)$ is said to be ω-*representable* if there is a digraph Δ such that the ω-end-poset of Δ is order isomorphic to $(X,\leq)$. A necessary condition for an order to be ω-representable is that every strictly increasing sequence has a supremum. Section 2.3 also contains a sufficient condition, but it is a slightly technical. From that sufficient condition, the fact that a *chain* is ω-representable if and only if every strictly increasing sequence has a supremum can be derived. Dual results are valid for ω^*-ends. Furthermore, some results concerning composed-end-representability are presented.

Section 2.4 is concerned with the search of simple sub- and superdigraphs of a digraph Δ which have the same end structure as Δ. A subdigraph Ξ of Δ is called *endfaithful with* Δ if Ξ contains a 1-track from every end of Δ and, whenever two 1-tracks belong to distinct ends of Ξ, they belong also to distinct ends of Δ. A characterization of those digraphs which are endfaithful with their symmetric (resp. transitive) hull is given. A vertex v of a digraph Δ is called an *initial vertex* of Δ if Δ contains a $v \to w$-track for every vertex w of Δ. It is proved that every countable digraph Δ with an initial vertex and no pair of $\leq$-comparable ω-ends contains an ω-endfaithful, spanning arborescence with the initial vertex v. This generalizes an old result of HALIN (Satz 3 in [29]).

The other sections of Chapter 2 are all devoted to the generalization of results about rays in undirected graphs to results about one-way infinite tracks in digraphs. It is shown in Section 2.2 that every digraph contains a system of disjoint ω-tracks of maximum cardinality, which generalizes Satz 1 of [32]. There exists an analogue of the characterization of rayless graphs given in [30] by HALIN for digraphs (Section 2.7). WATKINS [87] showed that in a vertex-transitive, infinite, connected, locally finite graph every vertex lies on a geodesic two-way infinite path. A generalization of this result for digraphs can be found in Section 2.5 where it is proved that every vertex of an infinite, connected, v-transitive digraph Δ lies on a two-way infinite track (which can be chosen to be geodesic if Δ is locally finite). The main result of Chapter 2 is a digraph-adapted version of the above mentioned theorem that every infinite, connected, vertex-transitive graph has 1, 2, or at least $2^{\aleph_0}$ ends and can be found at the end of Section 2.6. Among other things, it states that every vertex-transitive, connected digraph Δ such that the underlying undirected graph has exactly two ends is a directed strip, and that every directed strip has exactly two composed ends. (A *directed strip* is a connected, locally finite digraph Δ admitting a translation such that the underlying undirected graph has exactly two ends.) Further, it includes the statement that every vertex-transitive, connected, locally finite digraph such that the underlying undirected graph has at least $2^{\aleph_0}$ ends has at least $2^{\aleph_0}$ composed ends. Section 2.6 contains also some statements concerning translations of digraphs.

The second part of this thesis (Chapter 3) is occupied with the classification of all vertex-transitive, planar strips. One of the main parameters of the classification is κ_∞, the minimal number of vertices one has to remove from the graph to obtain at least two infinite components. Every vertex-transitive, planar strip Σ with $\kappa_\infty(\Sigma) \geq 3$ is 3-connected. Hence it has a unique embedding into the plane in the sense that the cyclic order of the edges around a vertex is the same at every vertex in every embedding — up to a possible simultaneous reversal of the cyclic edge orderings at all vertices. It follows that — apart from a finite number of exceptions with the connectivities 1 resp. 2 — every vertex-transitive, planar strip has a vertex type. It is possible to apply EULER's Formula to planar strips, and one easily derives that there are only eleven vertex types which an infinite, vertex-transitive, planar graph can have. Taking into account that Σ is a vertex-transitive strip, one can exclude five more vertex types, and it turns out that Σ must have one of the vertex types (3^6), $(3^3.4^2)$, $(3^2.4.3.4)$, (4^4), (6^3), and (4.8^2). If Σ is a planar strip of one of these vertex types with $\kappa_\infty(\Sigma) \geq 3$, then there is only a finite number (depending on κ_∞ and the vertex type) of possible structures for Σ. All these structures are determined, yielding that every planar strip of one of the vertex types (3^6), $(3^3.4^2)$, (4^4), (6^3), and (4.8^2) with $\kappa_\infty \geq 3$ is vertex-transitive. If Σ has the vertex type $(3^2.4.3.4)$, then there is only one family that is vertex-transitive whereas all others are not. Besides the exploitation of the properties of planar graphs that have a vertex type the classification also relies on the fact that a vertex-transitive, planar strip can be embedded on the surface of a 2-way infinite cylinder in a certain way. Further, a lemma is used which enables one to recognize end-separators of cardinality κ_∞ of a planar strip.

Versions of the Sections 2.1 and 2.3 will appear in Discrete Mathematics [94].

Chapter 1

Preliminaries

1.1 Set Theoretic Notation

The relations "proper subset of" ("subset of", "finite proper subset of", "finite subset of") are denoted by $\subset$ ($\subseteq$, $\sqsubset$, $\sqsubseteq$), respectively. The empty set is denoted by $\emptyset$. For the set of natural numbers and its n^{th} segments the following symbols shall be used: $\mathbb{N} := \{1,2,3,...\}$, $\mathbb{N}_0 := \{0,1,2,...\}$, $[n] := \{1,2,...,n\}$, $[0] := \emptyset$. The set of all integers, rational numbers, and real numbers will be denoted by $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$, respectively. If X is a set, then $|X|$ shall denote the cardinality of X. $|X| = \infty$ expresses the fact that $|X|$ is infinite. If ν is a cardinal, then $succ(\nu)$ denotes the smallest cardinal $> \nu$. The predecessor of a non-limit ordinal ν is denoted by $pred(\nu)$. If X and Y are sets, then $X \times Y := \{(x,y) \mid x \in X, y \in Y\}$ denotes the *Cartesian product of X and Y*. A *partition of the set X* is a set of pairwise disjoint non-empty subsets of X such that the union of all these sets yields X. Throughout this thesis, the symbol a may be shorthand for $\{a\}$.

1.2 Relations

If X and Y are sets, then $f \subseteq X \times Y$ is called a *map* or *function (from X into Y)* (written $f : X \to Y$) if, for every $x \in X$, there exists a unique $y_x \in Y$ such that $(x,y_x) \in f$. y_x is also denoted by $f(x)$. If $S \subseteq X$, then $f[S] := \{f(x) \mid x \in S\}$ denotes the *image of S under f*, and f may be shorthand for $f \cap (S \times Y)$. If $f : X \to Y$ and $f' : Y \to Z$ are maps, then $f \circ f'$ $:= \{(x, f'(f(x)) \mid x \in X\} \subseteq X \times Z$ is a map which is called the *composition of f and f'*. A map $f : X \to Y$ is called...
...*injective* if, for all $x, x' \in X$, $f(x) = f(x')$ implies $x = x'$.
...*surjective* if $f[X] = Y$.
...*bijective* if f is injective and surjective.

If X is a set and $R \subseteq X \times X$, then R is called a *relation on X*, and (X,R) denotes the set X with the relation R. Instead of $(x,y) \in R$, it is often written xRy. A relation R on X is called...
...*reflexive* if, for all $x \in X$, $(x,x) \in R$.
...*symmetric* if, for all $x,y \in X$, $(x,y) \in R$ implies $(y,x) \in R$.
...*antisymmetric* if, for all $x,y \in X$, $(x,y),(y,x) \in R$ implies $x = y$.
...*transitive* if, for all $x,y,z \in X$, $(x,y),(y,z) \in R$ implies $(x,z) \in R$.
...*total* if, for all $x,y \in X$, $(x,y) \in R$ or $(y,x) \in R$.

(X,R) (the relation R) is called a...
...*quasiordered set* or *quoset* (*quasiorder*), respectively, if R is a reflexive and transitive relation on X.
...*partially ordered set* or *poset* (*partial order*), respectively, if R is a reflexive, antisymmetric, and transitive relation on X. $x \in X$ is called a *least* (*greatest*) element of (X,R) if, for all $y \in X$, xRy (yRx), respectively.
...*totally ordered set* or *chain* (*total order*), respectively, if R is a partial order and, in addition, a total relation on X.
...*(dually) well-ordered set* or *(d)woset* (*(dual) well order*), respectively, if every non-empty subset of X contains a least (greatest) element, respectively. Every (d)woset is a chain.
...*R-partitioned set* (*equivalence relation*), respectively, if R is a reflexive, symmetric and transitive relation.

If $(X,\leq)$ is a quoset and $Y \subseteq X$, then $(Y,\leq)$ shall be used instead of $(Y, \leq \cap (Y \times Y))$, and if $x,y \in X$, then $x < y$ means that $x \leq y$ and $y \not\leq x$, and $x \geq y$ means that $y \leq x$. Further, two disjoint subsets Y, Y' of X are called *independent* if, for all $y \in Y$, $y' \in Y'$, neither $y \leq y'$ nor $y' \leq y$. If $(X,\leq)$ is a poset, then $x \in X$ is called the *supremum* (*infimum*) *of* $A \subseteq X$, if $a \leq x$ ($a \geq x$) holds for all $a \in A$ and, whenever $a \leq y \in X$ ($a \geq y \in X$) holds for all $a \in A$, then $y \geq x$ ($y \leq x$), respectively. $Y \subseteq X$ is called *upper* (*lower*) *bounded (in* $(X,\leq)$*)* if there exists $x \in X$ such that, for all $y \in Y$, $y \leq x$ ($x \leq y$), respectively. A *strictly increasing sequence* $(x_i)_{i\in\mathbb{N}}$ in $(X,\leq)$ is a sequence $x_1 < x_2 < x_3 < \ldots$ of elements of X. Two posets $(X,\leq)$, $(X',\leq')$ are called *(order) isomorphic* if there exists a bijective map $\varphi : X \to X'$ such that, for all $x,y \in X$, $x \leq y$ if and only if $f(x) \leq' f(y)$.

If R is a relation on the set X, then let, for every $x \in X$,
$R(x) := \{y \in X \mid (x,y) \in R\}$,
$X_R := \{R(y) \mid y \in X\}$,
$R^{-1} := \{(y,z) \mid (z,y) \in R\}$,
$\tilde{R} := R \cap R^{-1}$, and
$R^* := \{(\tilde{R}(x),\tilde{R}(y)) \mid \text{there are } x' \in \tilde{R}(x), y' \in \tilde{R}(y) \text{ such that } (x',y') \in R\}$.

Clearly, if R is an equivalence relation on X, then $\{R(x) \mid x \in X\}$ is a partition of X, and the $R(x)$, $x \in X$, are called the *classes (of R)*. The following is a very well known order theoretic result [ref. to [18], p.62, 3.19 Satz]:

Proposition 1.1.
Let (X,R) be a quoset. Then $\tilde{R}$ is an equivalence relation and R^* a partial order on $X_{\tilde{R}}$. □

ω denotes the order type of $\mathbb{N}$.

1.3 Algebraic Notation

A *group* $(G,\circ)$ is a set G containing an element e, the so-called *neutral element (of G)*, together with a map $\circ : G \times G \to G$ such that, for all $f,g,h \in G$, $\circ(\circ(f,g),h) = \circ(f,\circ(g,h))$ and, for all $g \in G$, $\circ((e,g)) = \circ((g,e)) = g$ and there exists $g^{-1} \in G$ such that $\circ((g,g^{-1})) = \circ((g^{-1},g)) = e$. Every group contains exactly one neutral element. $H \subseteq G$ is called a *subgroup (of G)* (in symbols: $H \leq G$) if $(H,\circ)$ is a group. Instead of $\circ((g,g')) = h$, it is written

$g \circ g' = h$. For every $g \in G$, let $g^0 := e$ and $g^i := g^{i-1} \circ g$ and $g^{-i} := (g^i)^{-1}$ ($i \in \mathbb{N}$). Then, for every $g \in G$, $< g > := \{g^i \mid i \in \mathbb{Z}\} \leq G$.

Let $(G, \circ)$ be a group with the neutral element e and X be a non-empty set. It is said that G *acts on* X if there exists a map $\bullet : G \times X \to X$ such that, for all $g, h \in G$ and $x \in X$, $\bullet((e,x)) = x$ and $\bullet((g \circ h, x)) = \bullet((g, \bullet((h,x))))$. It is said that G *acts transitively on* X if, for all x, $y \in X$, there exists a $g \in G$ such that $\bullet((g,x)) = y$. The relation $R_G := \{(x,y) \mid$ There exists $g \in G$ such that $\bullet((g,x)) = y\} \subseteq X \times X$ is an equivalence relation whose classes are called *orbits (of G in X (with respect to* $\bullet$*))*.

Let n, $m \in \mathbb{Z}$. The fact that n is (not) divisible by m is written $m \mid n$ ($m \nmid n$), respectively. m *mod* n denotes the unique integer $0 \leq k < n$ such that $m = cn + k$ for some $c \in \mathbb{Z}$.

1.4 (Di)Graphs, Sub-/Super(di)graphs, Minors, Subdivisions

A *graph* Γ is a set $V(\Gamma)$ — the *vertex set of* Γ — whose elements are called the *vertices (of* Γ*)*, together with a set $E(\Gamma)$ of 2-element subsets of $V(\Gamma)$ — the *edge set (of* Γ*)* — whose elements are called the *edges (of* Γ*)*, and it will be written $\Gamma = (V, E)$. A *directed graph* (or *digraph*) Δ is a set $V(\Delta)$ — the *vertex set of* Δ — whose elements are also called *vertices*, together with a relation $A(\Delta)$ on $V(\Delta)$ — the *arc set of* Δ — whose elements are called the *arcs (of* Δ*)*, and it will be written $\Delta = (V, A)$. The (di)graphs considered in this thesis are assumed to have no multiple edges (arcs), though, in digraphs, 2-circuits may be possible, i.e. $(u,v) \in A(\Delta)$ and $(v,u) \in A(\Delta)$. Throughout this thesis, uv will be shorthand for $\{u,v\}$ whenever $\{u,v\}$ is an edge of some graph.

If Δ and Δ' are (di)graphs and if there exists a bijective map $\varphi : V(\Delta) \to V(\Delta')$ such that, for all $u, v \in V(\Delta)$, $uv \in E(\Delta)$ ($(u,v) \in A(\Delta)$) if and only if $\varphi(u)\varphi(v) \in E(\Delta')$ ($(\varphi(u), \varphi(v)) \in A(\Delta')$), then Δ and Δ' are called *isomorphic*, and φ is called an *isomorphism (between* Δ *and* Δ'*)*. If $\Delta = \Delta'$, then φ is also called an *automorphism (of* Δ*)*. $Aut(\Delta)$ denotes the set of all automorphisms of a (di)graph Δ and is a group together with the composition of maps.

Two vertices u,v of a (di)graph Δ are called *adjacent* if $uv \in E(\Delta)$ ($(u,v) \in A(\Delta)$ or $(v,u) \in A(\Delta)$). If uv is an edge ((u,v) is an arc) of Δ, then it is said that the edge uv (the arc (u,v)) is *incident with the vertices u and v*.

Ξ is called a *sub(di)graph of* Δ and Δ is called a *super(di)graph of* Ξ (in symbols: $\Xi \subseteq \Delta$) if $V(\Xi) \subseteq V(\Delta)$ and $E(\Xi) \subseteq E(\Delta)$ ($A(\Xi) \subseteq A(\Delta)$) such that $E(\Xi)$ ($A(\Xi)$) does not contain an edge (arc) that is incident with a vertex $\notin V(\Xi)$. Sets of vertices, edges, and/or arcs may be identified with sub(di)graphs. If Ξ is a sub- or super(di)graph of Δ, then it is said that Ξ *spans* Δ if $V(\Xi) = V(\Delta)$. $\Xi \sqsubseteq \Delta$ means that Ξ is a finite sub(di)graph of Δ. If $\Xi \subseteq \Delta$, then $\Delta[\Xi]$ denotes the *induced sub(di)graph of* Δ *on* $V(\Xi)$, i.e. $V(\Delta[\Xi]) = V(\Xi)$ and $E(\Delta[\Xi])$ ($A(\Delta[\Xi])$) contains *all* edges (arcs) of Δ that are not incident with a vertex $\notin V(\Xi)$. $\Delta - \Xi$ denotes the induced sub(di)graph of Δ which is spanned by $V(\Delta) - V(\Xi)$.

Clearly, $Aut(\Delta)$ acts on the set of all sub(di)graphs of Δ. If $\varphi \in Aut(\Delta)$ and $\Xi \subseteq \Delta$, then $\varphi[\Xi]$ denotes the subgraph of Δ resulting from this action.

If $\mathcal{P}$ is a partition of the vertex set of the (di)graph Δ, then $\Delta_{\mathcal{P}}$ denotes the (di)graph which has $\mathcal{P}$ as its vertex set and where, for $S,T \in \mathcal{P}$, $S \neq T$, there is the edge ST (the arc (S,T)) if and only if there is an edge st (an arc (s,t)) in Δ with $s \in S$ and $t \in T$. Ξ is called a *minor of the graph* Γ if Ξ is isomorphic to a subgraph of some $\Gamma_{\mathcal{P}}$, where $\mathcal{P}$ is a partition of $V(\Gamma)$ such that, for every $P \in \mathcal{P}$, $\Gamma[P]$ is connected.

If Δ is a digraph, then...
...$\Delta^s := (V(\Delta), A(\Delta) \cup (A(\Delta))^{-1})$ is called the *symmetric hull of* Δ.
...Δ^t, the *transitive hull of* Δ, is the (uniquely determined) minimal spanning superdigraph of Δ such that $A(\Delta^t)$ is a transitive relation.
...$\Delta^* := \Delta_{\mathcal{VSC}}$, where $\mathcal{VSC}$ denotes the partition of $V(\Delta)$ defined by the strong components of Δ (see Section 1.7).
...Δ^u, the *underlying undirected graph of* Δ, is the graph with $V(\Delta^u) = V(\Delta)$ and $E(\Delta^u) = \{uv \mid u \neq v, (u,v) \in A(\Delta) \text{ or } (v,u) \in A(\Delta)\}$.

For the definition of paths and tracks see the next section. A (di)graph Δ^{sub} is called a *subdivision of* Δ if Δ^{sub} is isomorphic to some (di)graph that can be obtained from the (di)graph Δ by replacing every edge vw (arc (v,w)) of Δ by a v,w-path ($v \to w$-track) of length > 0 which has (except v and w) no vertex in common with Δ nor with any other "replacing path" ("replacing track"). Let Δ^{sub} be a subdivision of the (di)graph Δ, $\varphi : V(\Delta') \to V(\Delta^{sub})$ be an isomorphism between Δ' and Δ^{sub}, where Δ' is a (di)graph which has been obtained from Δ by replacing the edges (arcs) of Δ with paths (tracks), and $\psi : V(\Delta) \to V(\Delta')$ be defined by $\psi(v) = v$ for all $v \in V(\Delta)$. Then any vertex $\in \varphi[\psi[V(\Delta)]]$ is called a *main vertex of* Δ^{sub} *(with respect to* φ*)*.

1.5 Paths and Tracks, Cycles and Circuits. Reachability

A graph Γ is called a...
...*path* (*of length* n) if it is isomorphic to the graph Π with $V(\Pi) = \{v_0, v_1, ..., v_n\}$ and $E(\Pi) = \{v_{i-1}v_i \mid i \in [n]\}$ ($n \in \mathbb{N}_0$). The vertices of Γ that correspond to the vertices v_0 and v_n of Π are called the *rim vertices (of* Γ*)*.
...*cycle* (*of length* n) if it is isomorphic to the graph Π with $V(\Pi) = \{v_0, v_1, ..., v_{n-1}\}$ and $E(\Pi) = \{v_{i-1}v_i \mid i \in [n], \text{ indices mod } n\}$ ($n \in \{0\} \cup \{3,4,5,...\}$).
...*one-way infinite path* (or *ray*) if it is isomorphic to the graph Π with $V(\Pi) = \{v_i \mid i \in \mathbb{N}\}$ and $E(\Pi) = \{v_i v_{i+1} \mid i \in \mathbb{N}\}$. The vertex of Γ that corresponds to the vertex v_1 of Π is called the *rim vertex (of* Γ*)*.
...*two-way infinite path* (or *2-ray*) if it is isomorphic to the graph Π with $V(\Pi) = \{v_i \mid i \in \mathbb{Z}\}$ and $E(\Pi) = \{v_i v_{i+1} \mid i \in \mathbb{Z}\}$.

Two rays Π, Ψ of the graph Γ are called *(end-)equivalent* (in symbols: $\Pi \sim_\Gamma \Psi$) if there is a ray Φ which has in common infinitely many vertices both with Π and Ψ. The relation $\sim_\Gamma$ is an equivalence relation whose classes are called the *ends (of* Γ*)*. The set of all ends of Γ is denoted by Ω_Γ.

A digraph Δ is called a(n)...

...*zigzag-track* (*of length n*) if it is isomorphic to some graph Θ with $V(\Theta) = \{v_0, v_1, ..., v_n\}$ such that, for each $i \in [n]$, $A(\Theta)$ contains exactly one element of the set $\{(v_{i-1}, v_i), (v_i, v_{i-1})\}$ and nothing else ($n \in \mathbb{N}_0$). The vertices of Δ that correspond to the vertices v_0 and v_n of Θ are called the *rim vertices (of* Δ*)*.

...*track* (*of length n*) if it is isomorphic to the graph Θ with $V(\Theta) = \{v_0, v_1, ..., v_n\}$ and $A(\Theta) = \{(v_{i-1}, v_i) \mid i \in [n]\}$. The vertex of Δ that corresponds to the vertex v_0 (v_n) of Θ is called the *initial* (*terminal*) *vertex (of* Δ*)*, respectively.

...*circuit* (*of length n*) if it is isomorphic to the graph Θ with $V(\Theta) = \{v_0, v_1, ..., v_{n-1}\}$ and $A(\Theta) = \{(v_{i-1}, v_i) \mid i \in [n], \text{ indices mod } n\}$ ($n \in \mathbb{N}_0$).

...*1-zigzag-track* if it is isomorphic to some graph Θ with $V(\Theta) = \{v_i \mid i \in \mathbb{N}\}$ such that, for each $i \in \mathbb{N}$, $A(\Theta)$ contains exactly one element of the set $\{(v_{i+1}, v_i), (v_i, v_{i+1})\}$ (and nothing else). The vertex of Δ that corresponds to the vertex v_1 of Θ is called the *rim vertex (of* Δ*)*.

...ω-*track* if it is isomorphic to the graph Θ with $V(\Theta) = \{v_i \mid i \in \mathbb{N}\}$ and $A(\Theta) = \{(v_i, v_{i+1}) \mid i \in \mathbb{N}\}$.

...ω^*-*track* if it is isomorphic to the graph Θ with $V(\Theta) = \{v_i \mid i \in \mathbb{N}\}$ and $A(\Theta) = \{(v_{i+1}, v_i) \mid i \in \mathbb{N}\}$.

...*1-track* if it is either an ω- or an ω^*-track.

...*2-zigzag-track* if it is isomorphic to some graph Θ with $V(\Theta) = \{v_i \mid i \in \mathbb{Z}\}$ such that, for each $i \in \mathbb{Z}$, $A(\Theta)$ contains exactly one element of the set $\{(v_{i+1}, v_i), (v_i, v_{i+1})\}$ (and nothing else).

...*2-track* if it is isomorphic to the graph Θ with $V(\Theta) = \{v_i \mid i \in \mathbb{Z}\}$ and $A(\Theta) = \{(v_i, v_{i+1}) \mid i \in \mathbb{Z}\}$.

A zigzag-track is called *proper* if it is not a track. Note that every single vertex of a (di)graph can be interpreted as a path ((zigzag-)track) and also as a cycle (circuit) of length 0. $\mathcal{T}_\Delta(\infty)$ ($\mathcal{T}_\Delta(\omega)$, $\mathcal{T}_\Delta(\omega^*)$) denotes the set of all 1- (ω-, ω^*-) tracks of the digraph Δ, respectively. If Θ, Θ' are 1-tracks and are both ω- or both ω^*-tracks, then Θ and Θ' are said to have the *same direction*. Otherwise, Θ and Θ' are said to have *opposite directions*. Every infinite subpath (subtrack) of a path (1-track) Θ is called a *rest of* Θ. If Ξ is a sub(di)graph of the (di)graph Δ and Θ is a ray (1-track) of Δ, then, if it exists, $rest(\Theta - \Xi)$ denotes the maximum subray (sub-1-track) of Θ in $\Delta - \Xi$, respectively. A digraph Δ is called *acyclic* if it does not contain circuits of length > 0.

Let Θ be a (finite or infinite) path (track), respectively. Then, for $u, v \in V(\Theta)$, $\Theta[u, v]$ denotes the minimal subpath (subtrack) of Θ which includes both u and v, respectively. Further, $\Theta[u, v) := \Theta[u, v] - v$, $\Theta(u, v] := \Theta[u, v] - u$, and $\Theta(u, v) := \Theta[u, v] - u - v$. If Θ is a ray (1-track) belonging to the end [1] $\mathcal{E}$, then, for $v \in V(\Theta)$, $\Theta[v, \mathcal{E})$ denotes the minimal subray (sub-1-track) of Θ which contains v, respectively. Further, $\Theta(v, \mathcal{E}) := \Theta[v, \mathcal{E}) - v$. If Θ is a 2-track such that the ω-subtracks belong to an end $\mathcal{F}$ and the ω^*-subtracks belong to an end $\mathcal{E}$, then Θ is also called an $\mathcal{E} \to \mathcal{F}$*-track*. $\mathcal{E}, \mathcal{F}$*-rays* are defined in a similar manner. Further, subtracks (subpaths) of an $\mathcal{E} \to \mathcal{F}$-track ($\mathcal{E}, \mathcal{F}$-ray) like $\Theta[v, \mathcal{F})$ etc. are defined in the same way as the above subtracks (subpaths), respectively.

[1] For the definition of "directed" ends, see Section 2.1

Let Δ be a digraph and $\Xi, \Xi' \subseteq \Delta$. Every zigzag-track (track) with its one rim vertex (initial vertex) $\in \Xi$ and its other rim vertex (terminal vertex) $\in \Xi'$ and having no further vertex in common with $\Xi \cup \Xi'$ is called a Ξ,Ξ'*-zigzag-track* (a $\Xi \to \Xi'$*-track*), respectively. Ξ' is said *to be reachable from* Ξ (written $\Xi \to_\Delta \Xi'$) if there exists a $\Xi \to \Xi'$-track in Δ. $\Xi \leftrightarrow_\Delta \Xi'$ abbreviates the fact that Ξ and Ξ' are mutually reachable from each other. $V_\Delta(v \to)$ ($V_\Delta(\to v)$) denotes the set of all vertices that are reachable from v (resp. v is reachable from) and is called the *transmitting* (*receiving*) *area (of v)* ($v \in V(\Delta)$), respectively. Clearly, $(V(\Delta), \to_\Delta)$ is a quoset.

Let Θ be an arbitrary (finite or infinite) track. If $v \in V(\Theta)$, then v is said *to be on the track* Θ. If $\Xi, \Upsilon \subseteq \Theta$, $\Xi \cap \Upsilon = \emptyset$, and $v \to_\Theta w$ holds for all $v \in V(\Xi)$, $w \in V(\Upsilon)$, then it is said that "Ξ *is before* Υ *on* Θ" or "Υ *is behind* Ξ *on* Θ".

Two paths (tracks) Π, Π' in a (di)graph Δ are called *inner disjoint* if, apart from their rim vertices, Π and Π' are disjoint.

Remark 1.2.
If Δ is a digraph, then every (finite or infinite) path of Δ^u corresponds to an (equivalence)class of zigzag-tracks of Δ which all have the same vertices in the same arrangement. The set of all these classes is a partition of the set of all zigzag-tracks of Δ and can be mapped bijectively to the set of all paths of Δ^u, so that there is hardly any difference between running along the zigzag-tracks without consideration of the direction of the affiliated arcs and examining the underlying undirected paths. □

1.6 Vertices and Local Structures

A vertex w of the graph Γ is called a *neighbour of the vertex v (of* Γ*)* if $wv \in E(\Gamma)$. The *degree of a vertex v (of* Γ*)* is the cardinality of the set of all neighbours of v in Γ. Γ is called *locally finite* if all its vertices have a finite degree. If ν is some cardinal, then Γ is called ν*-regular* if all its vertices have the degree ν. If Ξ, $\Upsilon \subseteq \Gamma$, then $dist(\Xi, \Upsilon)$, the *distance of* Ξ *and* Υ *(in* Γ*)*, is the length of a shortest Ξ,Υ-path in Γ if it exists. If such a path does not exist, then the distance is defined to be infinite.

A vertex w of the digraph Δ is called an *inneighbour* (*outneighbour, neighbour*) of a vertex v of Δ if $(w,v) \in A(\Delta)$ ($(v,w) \in A(\Delta)$, $(w,v) \in A(\Delta)$ or $(v,w) \in A(\Delta)$), respectively. The *indegree* (*outdegree*, *degree*) *of a vertex v of* Δ is the cardinality of the set of all inneighbours (outneighbours, neighbours) of v in Δ, respectively. Δ is called *inner locally finite* (*outer locally finite*, *locally finite*) if all its vertices have a finite indegree (outdegree, degree), respectively. If ν and μ are cardinals, then Δ is called (ν,μ)*-regular* if every vertex of Δ has the indegree ν and the outdegree μ.

A vertex v of the digraph Δ is called an *initial vertex* (a *terminal vertex*) *of* Δ if $V_\Delta(v \to) = V(\Delta)$ ($V_\Delta(\to v) = V(\Delta)$), respectively.

1.7 Connectivity and Separation

A graph Γ is called *connected* if, for every $u,v \in V(\Gamma)$, Γ contains a u,v-path. A *component of* Γ is a maximal, connected subgraph of Γ. Let $n \in \mathbb{N}$. Γ is called *n-connected* if Γ has at least $n+1$ vertices and, for every pair $u,v \in V(\Gamma)$, there exist at least n pairwise inner disjoint u,v-paths in Γ. The *connectivity (of* Γ*)* is the minimum cardinal $\kappa(\Gamma)$ such that Γ is $\kappa(\Gamma)$-connected but not $succ(\kappa(\Gamma))$-connected.

A subgraph Ξ of Γ is said to *separate* Γ or to be a *separator of* Γ if $\Gamma - \Xi$ has at least two components. Let $\mathcal{E}$ and $\mathcal{F}$ be ends of Γ. It is said that Ξ *separates* $\mathcal{E}$ *and* $\mathcal{F}$ *(in* Γ*)* or that Ξ is an $\mathcal{E},\mathcal{F}$*-separator* if, for every $\Pi \in \mathcal{E}$, $\Psi \in \mathcal{F}$, Π and Ψ are in different components of $\Gamma - \Xi$. If Ξ is an edge set and Ξ separates Γ ($\mathcal{E}$ and $\mathcal{F}$), then Ξ is also called an *edge-separator* ($\mathcal{E},\mathcal{F}$*-edge-separator*) *of* Γ, respectively. A(n) (edge-) separator of cardinality ν is also called a ν*-(edge)-separator*.

The following terminology stems mainly from [49] and [50]. If Y is a subset of $V(\Gamma)$, then ∂Y, the *boundary of* Y *(in* Γ*)*, is defined to be the set of vertices $\notin Y$ that are adjacent to some vertex in Y. Further, $\overline{Y}$ denotes the set $V(\Gamma) \setminus Y \setminus \partial Y$. For the purpose of investigating the connectivity of infinite graphs, it is worth to examine the following parameters:

$$\kappa_\infty(\Gamma) \;:=\; min\{|\partial Y| \mid Y \text{ and } \overline{Y} \text{ are both infinite subsets of } V(\Gamma)\}$$

$$\lambda_\infty(\Gamma) \;:=\; min\{|F| \mid F \subseteq E(\Gamma) \text{ and } \Gamma - F \text{ has at least two infinite components}\}$$

If the corresponding minimum is taken over an empty set, then, by convention, this fact is written as $\kappa_\infty(\Gamma) = \infty$.

A subset Y of $V(\Gamma)$ is called an *(infinite) fragment of* Γ if Y and $\overline{Y}$ are both infinite and $|\partial Y| = \kappa_\infty(\Gamma)$ is finite.

A digraph Δ is called...

...*(weakly) connected* if Δ^u is connected.
...*unilaterally connected* if, for all $u,v \in V(\Delta)$, u is reachable from v or v is reachable from u.
...*strongly connected* if, for all $u,v \in V(\Delta)$, $u \leftrightarrow_\Delta v$.

A *(weak)* (*unilateral*, *strong*) *component of* Δ is a maximum, (weakly) (unilaterally, strongly) connected subdigraph of Δ, respectively. Clearly, the set consisting of every vertex set of a strong component of Δ is a partition of $V(\Delta)$.

Two subdigraphs Ξ, Υ of the digraph Δ are called *(strongly) separated in* Δ if no weak component of Δ contains vertices of both Ξ and Υ. Further, Υ is said to be *weakly separated from* Ξ *(in* Δ*)* if Υ is not reachable by Ξ.

1.8 Miscellaneous

A (di)graph Δ is called *v-transitive* (*e-transitive* (*a-transitive*)) if $Aut(\Delta)$ acts transitively on $V(\Delta)$ ($E(\Delta)$ ($A(\Delta)$)), respectively.

A k-regular spanning subgraph of Γ is also called a *k-factor (of* Γ*)*.

Let m, $n \in \mathbb{N}$ and Γ be a graph. Γ is called a K_n if it is isomorphic to a graph having n vertices and containing all possible edges between these vertices. Γ is called a $K_{m,n}$ if it is isomorphic to the graph Γ' with $V(\Gamma') = \{v_i, w_j \mid i \in [m], j \in [n]\}$ and $E(\Gamma') = \{v_i w_j \mid i \in [m], j \in [n]\}$.

A connected graph without cycles is called a *tree*. An *arborescence* is an acyclic digraph Λ such that Λ^u is a tree, and which has an initial vertex. If Λ is an arborescence all of whose vertices have the outdegree 2, then Λ is called a *dyadic arborescence*.

Let Γ be a graph. The *diameter (of* Γ*)* (in symbols: $diam(\Gamma)$) is defined as follows:

$$diam(\Gamma) := \begin{cases} max\{dist(x,y) \mid x,y \in V(\Gamma)\} & \text{, if the maximum exists} \\ 0 & \text{, if } V(\Gamma) = \emptyset \\ \infty & \text{, otherwise.} \end{cases}$$

In a designation of the form X_Δ or $X_\Delta(Y)$ the parameter Δ may be suppressed whenever there is no danger of confusion. If it is said that some graphs are disjoint this always means that these graphs are vertex-disjoint.

Chapter 2

An End Concept for Digraphs

2.1 Definition and Basic Properties

The relation $\leq_\Delta$ on $\mathcal{T}_\Delta(\infty)$ mentioned in the introduction is fundamental to all further investigations in this chapter. Hence, its definition will be repeated here, followed by some basic statements about it.

Definition 2.1 ($\leq_\Delta$, $\sim_\Delta$).
If Θ, Θ' are 1-tracks of the digraph Δ, then $\Theta \leq_\Delta \Theta'$ expresses the fact that there are infinitely many $\Theta \to \Theta'$–tracks in Δ, and $\Theta \sim_\Delta \Theta'$ means that $\Theta \leq \Theta'$ and $\Theta' \leq \Theta$. □

Clearly, if $\Theta \leq \Psi$, then $\Theta' \leq \Psi'$ for all rests Θ' of Θ, Ψ' of Ψ.

The following proposition is helpful in many contexts to simplify proofs and can easily be verified:

Proposition 2.2.
Let Δ be a digraph. Then the following statements are equivalent:

(1) $\Theta \leq \Psi$.

(2) $rest(\Theta - S) \to_{\Delta - S} rest(\Psi - S)$ *for all* $S \sqsubseteq V(\Delta)$. □

The proof of the next result is also trivial, but it shows that it makes no sense to introduce an end concept based upon 1-zigzag-tracks since transitivity cannot be expected. Of course, similar statements hold for $(\mathcal{T}(\omega), \leq)$ and $(\mathcal{T}(\omega^*), \leq)$.

Proposition 2.3.
Let Δ be a digraph. Then $(\mathcal{T}(\infty), \leq)$ is a quoset. □

Proof:
Clearly, $\leq$ is reflexive since every vertex itself is a track. The relation $\leq$ is also transitive: Let Θ and Π be 1-tracks, Ψ be an ω-track, $\Theta \leq \Psi$, $\Psi \leq \Pi$, and $S \sqsubseteq V(\Delta)$. According to Proposition 2.2, there exists a $rest(\Theta - S) \to rest(\Psi - S)$–track Λ in $\Delta - S$ and a $rest(\Psi - (S \cup \Lambda)) \to rest(\Pi - (S \cup \Lambda))$–track Λ' in $\Delta - (S \cup \Lambda)$. Now, $rest(\Theta - S) \to_{\Delta - S} rest(\Pi - S)$ can easily be seen, as the initial vertex of Λ' is behind the terminal vertex of Λ on Ψ. A similar construction is feasible if Ψ is an ω^*-track. The only difference is that the tracks Λ and Λ' have to be constructed in the reverse order. □

Now, Proposition 1.1 yields that $\sim_\Delta$ is an equivalence relation on $\mathcal{T}(\infty)$ (and also on $\mathcal{T}(\omega)$, $\mathcal{T}(\omega^*)$). Moreover, Proposition 1.1 yields that

$$\mathcal{E} \leq_\Delta \mathcal{E}' \quad :\Leftrightarrow \quad \text{there exist } \Theta \in \mathcal{E}, \Theta' \in \mathcal{E}' \text{ such that } \Theta \leq \Theta'$$

is a partial order on the $\sim$-factorization of $\mathcal{T}(\infty)$ (and also of $\mathcal{T}(\omega)$, $\mathcal{T}(\omega^*)$), which will be investigated in Section 3.

Definition 2.4 (directed ends).
Let Δ be a digraph. An equivalence class of $\mathcal{T}(\infty)$ with respect to $\sim$ is called a *composed end* or briefly a *c-end of* Δ. Ω_Δ*(c)* denotes the set of all c-ends of Δ. If Θ is a 1-track, then $\mathcal{E}_\Delta(c,\Theta)$ denotes the Θ-containing c-end of Δ. ω-*ends*, ω^*-*ends*, $\Omega_\Delta(\omega)$, $\Omega_\Delta(\omega^*)$, $\mathcal{E}_\Delta(\omega,\Theta)$, and $\mathcal{E}_\Delta(\omega^*,\Theta)$ are defined analogously. ω- and ω^*-ends are also called *simple ends*. If $\mathcal{E}$ is an arbitrary end of Δ, $\mathcal{E}^u$ denotes the set of all rays in Δ^u that exactly correspond to tracks in $\mathcal{E}$. If there are statements valid for each of the end-notions introduced here, then the symbols Ω_Δ etc. may be used in a consistent way. In other words, within one statement, the symbols Ω_Δ etc. can be substituted uniformly by $\Omega_\Delta(c)$, $\Omega_\Delta(\omega)$, and $\Omega_\Delta(\omega^*)$ etc. □

The concept of ω-ends catches the *forward* ω-ramification structure and the dual concept of ω^*-ends, the *backward* ω-ramification structure of a digraph Δ. On the other hand, the concept of c-ends allows a general view over the whole ω-ramification structure of Δ. Clearly, $|\Omega(c)| > 0 \Leftrightarrow \mathcal{T}(\infty) \neq \emptyset$, $|\Omega(\omega)| > 0 \Leftrightarrow \mathcal{T}(\omega) \neq \emptyset$, and $|\Omega(\omega^*)| > 0 \Leftrightarrow \mathcal{T}(\omega^*) \neq \emptyset$.

The following statement about the relations between composed and simple ends can easily be verified and justifies in a way the expression "composed ends".

Proposition 2.5.
Every simple end is completely contained in a composed end. A c-end is either an ω*-end, an* ω^**-end, or the union of exactly one* ω*-end and exactly one* ω^**-end. Hence,*
$|\Omega(\omega)| \leq |\Omega(c)|$,
$|\Omega(\omega^*)| \leq |\Omega(c)|$, *and*
$|\Omega(c)| \leq |\Omega(\omega)| + |\Omega(\omega^*)| \leq 2 \cdot |\Omega(c)|$. □

One has to be very careful in generalizing results about ends in undirected graphs to results about ends in digraphs. HALIN [29] defined two rays Π and Π' of an undirected graph Γ to be equivalent if there is a ray Π'' that meets both Π and Π' infinitely many times. If, for $\Theta, \Theta' \in \mathcal{T}_\Delta(\infty)$, equivalence had been stated as

$\Theta \sim'_\Delta \Theta'$:$\Leftrightarrow$ there is $\Theta'' \in \mathcal{T}_\Delta(\infty)$ which meets both Θ and Θ' infinitely many times,

then $\sim'$ would not necessarily have been an equivalence relation. It is not difficult to construct digraphs such that $\sim'$ is not transitive. The next proposition (compare with [29], 1.1) shows that every counterexample contains 1-tracks of opposite directions.

Proposition 2.6.
Let Δ *be a digraph and* $\Theta, \Psi \in \mathcal{T}(\infty)$*. Then the following statements are equivalent:*

(1) If Θ *and* Ψ *have the same direction, then* $\Theta \sim' \Psi$*. Otherwise, there are infinitely many pairwise disjoint circuits* $\subseteq \Delta$ *such that each of them has at least one vertex in common both with* Θ *and* Ψ*.*

(2) $\Theta \leftrightarrow_{\Delta - S} \Psi$ *for all* $S \sqsubseteq \Delta$.

(3) $\Theta \sim \Psi$. □

Proof:
(2) ⇔ (3) follows immediately from Proposition 2.2.
(1) ⇒ (3) is trivial.
(3) ⇒ (1) can be shown by straightforward inductive constructions. □

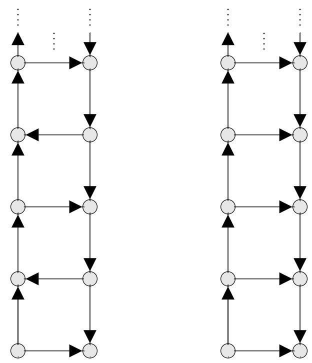
One obtains no new relation by replacing $\mathcal{T}(\infty)$ by $\mathcal{T}(\omega)$ or by $\mathcal{T}(\omega^*)$ in the definition of $\sim'$. If $\Theta, \Psi \in \mathcal{T}(\omega)$ and $\Theta \sim' \Psi$, then Proposition 2.6 shows that there is an ω-track Π which meets both Θ and Ψ infinitely many times. Thus, it is impossible that there is $\Pi \in \mathcal{T}(\omega^*)$ but no $\Pi' \in \mathcal{T}(\omega)$ which meets both Θ and Ψ infinitely many times. A dual statement holds if $\Theta, \Psi \in \mathcal{T}(\omega^*)$. Note that it would have been possible to define two rays Θ, Ψ in an undirected graph Γ to be equivalent if there are infinitely many pairwise disjoint cycles such that each of them has at least one vertex in common both with Θ and Ψ.

The next statement can be obtained by straightforward inductive constructions, too.

Proposition 2.7.
Let Δ *be a digraph. Then the following statements are valid.*

(1) If $\mathcal{E}$ *is a c-end containing two 1-tracks* Θ *and* Ψ *with opposite directions, then* Θ *and* Ψ *have infinitely many vertices in common or* Δ *contains a subdivision of the digraph in Fig. 2.1a). Similar statements hold for 1-tracks with the same direction in* ω*- and* ω^**-ends.*

(2) If $\mathcal{E}$, $\mathcal{F}$ *are two different comparable ends of* Δ, $\Theta \in \mathcal{E}$ *is an* ω*-track, and* $\Psi \in \mathcal{F}$ *is an* ω^**-track, then* Δ *contains a subdivision of the digraph in Fig. 2.1b). Similar statements hold for other combinations of directions of* Θ *and* Ψ. □



a) b)

Figure 2.1: see Proposition 2.7

The next result says something about the relations between ends in graphs and digraphs.

Proposition 2.8.
Let Δ be a digraph and $\mathcal{E}$, $\mathcal{F}$ ends of Δ. Then the following statements are valid.

(1) All rays $\in \mathcal{E}^u$ are contained in a single end of Δ^u.

(2) If $\mathcal{E}$ and $\mathcal{F}$ are comparable, then all rays in $\mathcal{E}^u$ and $\mathcal{F}^u$ are in the same end of Δ^u. Hence, every component of $(\Omega_\Delta, \leq)$ belongs to one end of Δ^u. On the other hand, an end of Δ^u can contain several components of $(\Omega_\Delta, \leq)$. Moreover, if $(\Omega_\Delta(c), \leq)$ is a chain, Δ^u has exactly one end.

(3) All rays in $\mathcal{E}^u$ and $\mathcal{F}^u$ are in the same end of Δ^u if and only if there exist $\Theta \in \mathcal{E}$ and $\Psi \in \mathcal{F}$ such that there are infinitely many pairwise disjoint Θ, Ψ-zigzag-tracks.

(4) Let $\Theta, \Psi \in \mathcal{T}_\Delta(\infty)$. Then Θ^u and Ψ^u are in the same end of Δ^u if and only if exactly one of the following statements holds:

(1) $\Theta \sim \Psi$.
(2) $\Theta < \Psi$.
(3) $\Psi < \Theta$.
(4) $\Theta \not\lesssim \Psi$ and $\Psi \not\lesssim \Theta$ and there exist infinitely many pairwise disjoint Θ, Ψ-zigzag-tracks. □

Proof:
Remark 1.2 yields (3). (1), (2), and (4) follow immediately from (3). □

Proposition 2.8 clarifies a fundamental difference between ends in graphs and ends in digraphs: Whereas two ends $\mathcal{E}, \mathcal{E}'$ of an undirected graph Γ can always be separated by a finite subgraph Ξ, this is not true for digraphs. In light of this, difficulties in obtaining interesting results about ends in digraphs with the aid of the separation of ends can be expected. Furthermore, Proposition 2.8 shows that the end-concept for digraphs investigated here is a refinement of certain subsets of ends of undirected graphs.

The following examples (see Fig. 2.2) give a flavour of the differences between “directed” and “undirected” ends: Consider the $\mathbb{N} \times \mathbb{N}$-grid Γ. It has exactly one end. Δ_2 is an end-less orientation of Γ. On the other hand, the upwards directed $\mathbb{N} \times \mathbb{N}$-grid Δ_1 has two (completely independent) strictly increasing sequences of ω-ends and one ω-end — consisting of all “diagonal” ω-tracks that are unbounded in both coordinates (the fat ω-track shows such a “diagonal” ω-track) — as the supremum of the two strictly increasing sequences. In Fig. 2.2, the ω-ends of Δ_1 are drawn as ideal points.

In the sequel, the relations between ends and the strong components of a digraph are studied.

Lemma 2.9.
Let Δ be a digraph, $\mathcal{E}$ be an end (simple or composed) of Δ, and Ξ be a strong component of Δ. If there exists $\Theta \in \mathcal{E}$ such that $|\Theta \cap \Xi| = \infty$, then Ξ contains a rest of every $\Psi \in \mathcal{E}$. In this sense, one can say that an end $\mathcal{E}$ of Δ belongs to a strong component of Δ. If an end $\mathcal{E}$ belongs to no strong component of Δ, then every $\Psi \in \mathcal{E}$ intersects with infinitely many strong components of Δ. □

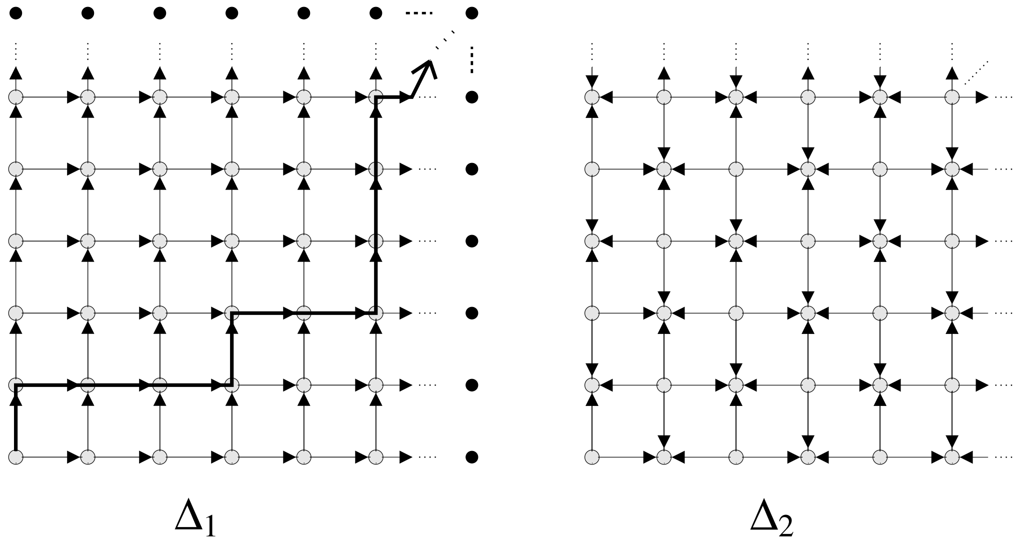


Figure 2.2: Some orientations of the $\mathbb{N} \times \mathbb{N}$-grid

Proof:
Let Ξ be a strong component of Δ and $\Theta \in \mathcal{E}$ such that $|\Theta \cap \Xi| = \infty$. Ξ contains a rest of Θ. If $\Psi \in \mathcal{E}$, then $v \leftrightarrow_\Delta \Xi$ follows from Proposition 2.6 for all but finitely many vertices v of Ψ. Hence, Ξ contains a rest of Ψ. $\square$

Lemma 2.9 motivates the following definition:

Definition 2.10 (cyclic and acyclic ends).
A 1-track Θ (a (simple or composed) end $\mathcal{E}$) of a digraph Δ is called *cyclic* if it belongs to a strong component in the sense of Lemma 2.9. Otherwise, the 1-track Θ (the end $\mathcal{E}$) is called *acyclic*. $\mathcal{T}_\Delta^{ac}(\omega)$ ($\Omega_\Delta^{ac}(\omega)$) denotes the set of all acyclic ω-tracks (ω-ends) of Δ, respectively. $\mathcal{T}_\Delta^{ac}(\omega^*)$, $\Omega_\Delta^{ac}(c)$ etc. are defined analogously. $\square$

The following proposition deals with the inner structure of acyclic ends.

Proposition 2.11.
An acyclic composed end of the digraph Δ contains either exclusively ω-tracks or exclusively ω^*-tracks. Hence, there are no acyclic proper composed ends. Furthermore, if all ends of Δ are acyclic, then $|\Omega_\Delta(c)| = |\Omega_\Delta(\omega)| + |\Omega_\Delta(\omega^*)|$.

Proof:
Let $\mathcal{E}$ be an acyclic c-end of Δ, $\Theta \in \mathcal{E}$ be an ω-track, and $\Psi \in \mathcal{E}$ be an ω^*-track. Let v be a vertex on Θ reachable from Ψ and M the set of all vertices on Θ behind v. It follows $v \leftrightarrow u$ for all $u \in M$, since Ψ is an ω^*-track and $\Theta, \Psi \in \mathcal{E}$. Therefore, $\Theta[M]$ belongs to a strong component of Δ, contradicting the acyclicity of $\mathcal{E}$. $\square$

For the purpose of studying the relations between the end structure of Δ and Δ^* some new terminology is needed. $\mathcal{S}_\Delta$ denotes the set of all subdigraphs of Δ. For $\Lambda \subseteq \Delta$, let $\Lambda(\Delta^*)$ be the subdigraph of Δ^* which has exactly those strong components of Δ as vertices that intersect with Λ and which contains an arc (Ξ,Ξ') if there is an arc from a vertex of Ξ to a vertex of Ξ' in Λ. Clearly,

$$\varphi^* : \mathcal{S}_\Delta \to \mathcal{S}_{\Delta^*},\ \varphi^*(\Lambda) := \Lambda(\Delta^*),$$

is a surjective mapping.

Proposition 2.12.
Let Δ be a digraph. Then $\varphi^*|_{\mathcal{T}^{ac}_\Delta(\omega)} : \mathcal{T}^{ac}_\Delta(\omega) \to \mathcal{T}_{\Delta^*}(\omega)$ is a surjective mapping. A dual statement holds for ω^*-tracks. □

Proof:
If Θ is an acyclic ω-track of Δ then $\Theta(\Delta^*) \in \mathcal{T}_{\Delta^*}(\omega)$, since, for every strong component Ξ of Δ which intersects Θ, $\Xi \cup \Theta$ is a finite subtrack of Θ. Now, let Θ be an ω-track of Δ^*. Starting from the initial vertex, let the vertices of Θ (strong components of Δ !) be denoted by Ξ_1, Ξ_2, Ξ_3,... . For $i \in \mathbb{N}$, there exists an arc (a_i, b_i) such that $a_i \in \Xi_i$ and $b_i \in \Xi_{i+1}$. Furthermore, in Ξ_{i+1}, there exists a $b_i \to a_{i+1}$-track Θ_i. Since $\Xi_i \cap \Xi_j = \emptyset$ for $i \neq j$, the union of the Θ_i and (a_i, b_i) $(i \in \mathbb{N})$ is an ω-track $\Theta' \subseteq \Delta$. Obviously, $\Theta'(\Delta^*) = \Theta$. □

Lemma 2.13.
Let Δ be a digraph and Θ, Ψ be 1-tracks of Δ. Θ and Ψ are contained in the same acyclic end of Δ if and only if $\Theta(\Delta^*)$ and $\Psi(\Delta^*)$ are contained in the same end of Δ^*. □

Proof:
$\Rightarrow$: Let $\mathcal{E}$ be an acyclic ω-end of Δ and $\Theta, \Psi \in \mathcal{E}$. For every infinite system $\mathcal{T}$ of pairwise disjoint tracks joining Θ and Ψ and every strong component S of Δ, S intersects only with finitely many tracks $\in \mathcal{T}$ (Assume $(\Pi_i)_{i\in\mathbb{N}}$ to be a family of pairwise disjoint $\Theta \to \Psi$–tracks such that, for all $i \in \mathbb{N}$, Π_i intersects with the strong component Ξ, and let v a terminal vertex of one of the Π_i on Ψ. Then, of course, all vertices v' on Ψ behind v are reachable from Ξ. On the other hand, Θ, and hence Ξ, would be reachable from all vertices of Ψ. Hence, a rest of Ψ would belong to Ξ, a contradiction.) Therefore, if we remove finitely many strong components from Δ, then Θ and Ψ remain mutually reachable from each other. Hence, for all $\Upsilon' \sqsubseteq \Delta^*$, $\Theta(\Delta^*) \leftrightarrow_{\Delta^*-\Upsilon'} \Psi(\Delta^*)$. Because of Proposition 2.6, $\Theta(\Delta^*)$ and $\Psi(\Delta^*)$ belong to the same ω-end of Δ^*. A dual argumentation holds for ω^*-tracks. Because of Proposition 2.11 there is nothing left to show.
$\Leftarrow$: If $\Theta(\Delta^*)$ and $\Psi(\Delta^*)$ are contained in the same ω-end of Δ^*, then Θ and Ψ are acyclic. Furthermore, for any $\Xi \sqsubseteq \Delta$, $\Xi(\Delta^*)$ is finite. Proposition 2.6 yields $\Theta(\Delta^*) \leftrightarrow_{\Delta^*-\Xi(\Delta^*)} \Psi(\Delta^*)$. Now, $\Theta \leftrightarrow_{\Delta-\Xi} \Psi$ can be shown with the method in the proof of Proposition 2.12. □

Theorem 2.14.
Let Δ be a digraph. $\varphi : \Omega^{ac}_\Delta \to \Omega_{\Delta^*}$, $\varphi(\mathcal{E}) := \mathcal{E}^*$, where $\mathcal{E}^*$ denotes the end of Δ^* which contains $\Theta(\Delta^*)$ for some $\Theta \in \mathcal{E}$, is a bijective mapping. In other words: There exists a natural bijection between the acyclic ends of Δ and the ends of Δ^*. □

Proof:
That φ is well-defined follows from Lemma 2.13 '$\Rightarrow$'. The injectivity of φ follows from Lemma 2.13 '$\Leftarrow$'. The surjectivity of φ can be derived from Proposition 2.12. □

Corollary 2.15.
$|\Omega_{\Delta^*}| \leq |\Omega_\Delta|$ with equality in the finite case if and only if no strong component of Δ has an end. □

2.2 Thickness of Ends

There are several ways to define a measure for the thickness of ends. HALIN [33] showed that every end of an undirected graph Γ contains a system of pairwise disjoint rays of maximum cardinality and defined the thickness $m_1(\mathcal{E})$ of an end $\mathcal{E}$ of a graph Γ to be this maximum cardinality. It seems to be impossible to carry over the proof of HALIN's result to a directed version with only some minor modifications (if the directed version of this result is true at all). The problem is that the locally finite graph H constructed in HALIN's proof would not necessarily be one-ended in the directed case. A (positive or negative) result concerning this question would be interesting:

Problem 2.
Does every ω-end of a digraph Δ contain a system of pairwise disjoint ω-tracks of maximum cardinality? □

An end $\mathcal{E}$ of a (di)graph Γ is called *thick* if $m_1(\mathcal{E}) \geq \aleph_0$. Otherwise, $\mathcal{E}$ is called *thin*. Another measure for the thickness of ends based on the distance metric of the graph ("*fat*" and "*slim*" ends) has been introduced by PICARDELLO and WOESS [60] but shall not be discussed here. Several results concerning both measures can be found in a paper of THOMASSEN and WOESS [80].

Although it seems to be a difficult problem to show whether every end of a digraph contains a system of pairwise disjoint 1-tracks of maximum cardinality or not, it is possible to generalize another related result of HALIN to digraphs. In [32] HALIN proved that, in an undirected graph Γ, the maximum number of pairwise disjoint rays always exists. A similar statement is true for digraphs as has been shown in the subsequent theorem. The proof of Theorem 2.17 is very similar to the proof in the undirected case. Hence it will be possible to refer very often to the proof of HALIN in [32], which uses a variant of MENGERs Theorem for infinite paths in locally finite graphs (Satz 3 in [28]). Before proving the main result of this section, Satz 3 of [28] is restated here in a digraph-adapted form for ω-tracks:

Lemma 2.16.
Let Δ be an outer locally finite digraph, $v \in V(\Delta)$, and $n \in \mathbb{N}$. Then the following statements are equivalent:

(1) *There exist n ω-tracks with the initial vertex v in Δ that, apart from v, are pairwise disjoint.*

(2) *If one removes arbitrarily $n-1$ vertices $\neq v$ from Δ, then in the remaining digraph there still exists an ω-track with the initial vertex v.*

(3) *If one removes arbitrarily $n-1$ vertices $\neq v$ from Δ, then v has an infinite transmitting area.* □

Proof: Can just be proved as in [28]. One has only to replace some graph concepts by the corresponding digraph concepts and to apply an analogue of MENGERs Theorem for digraphs which can be found in [86], p.69. □

Theorem 2.17.
Every digraph Δ contains a system of pairwise disjoint ω-tracks of maximum cardinality. □

Proof:
For a digraph Δ, let $\mathcal{T}_\Delta(\omega;\mu)$ denote the set of all systems of cardinality μ of pairwise disjoint ω-tracks in Δ. If $\mathcal{T}_\Delta(\omega;\aleph_1) \neq \emptyset$ or there is $n \in \mathbb{N}_0$ such that $\mathcal{T}_\Delta(\omega;n) \neq \emptyset \wedge \mathcal{T}_\Delta(\omega;n+1) = \emptyset$ it is possible to argue similar as in the proof of [32], Satz 1. Hence it remains the case that, for all $n \in \mathbb{N}$, $\mathcal{T}_\Delta(\omega;n) \neq \emptyset$. It has to be shown that $\mathcal{T}_\Delta(\omega;\aleph_0) \neq \emptyset$.

1. Δ can be assumed to be locally finite.

Can be shown as in [32].

2. It can be assumed that Δ is connected and, for every $S \sqsubseteq V(\Delta)$, $\Delta - S$ has only one component containing ω-tracks. Further, it can be assumed that every vertex of Δ lies on an ω-track.

Suppose Δ to have infinitely many ω-ends $\mathcal{E}_i$ $(i \in \mathbb{N})$. Choose some $\Theta_1 \in E_1$. If $\Theta_1 \in \mathcal{E}_1$, ..., $\Theta_n \in \mathcal{E}_n$ have already been chosen such that Θ_1,...,Θ_n are pairwise disjoint, then it is possible to find some $\Theta_{n+1} \in \mathcal{E}_{n+1}$ which has no vertex in common with $\bigcup_{i\in[n]} \Theta_i$, since any $\Theta \in \mathcal{E}_{n+1}$ has only finitely many vertices in common with $\bigcup_{i\in[n]} \Theta_i$. Hence it is possible to find inductively a system of infinitely many pairwise disjoint ω-tracks.

If Δ has only finitely many ω-ends, then there must be one ω-end $\mathcal{E}$ that contains n pairwise disjoint ω-tracks for every $n \in \mathbb{N}$. For every $i \in \mathbb{N}$, let $\{\Psi_{i,1},..., \Psi_{i,i}\}$ be a system of pairwise disjoint ω-tracks $\in \mathcal{E}$. For every $i,j,k,l \in \mathbb{N}$ such that $j \leq i$, $k \leq l$, and $(i,j) \neq (k,l)$ let $(\Pi_{m,i,j,k,l})_{m\in\mathbb{N}}$ be a system of pairwise disjoint $\Psi_{i,j} \to \Psi_{k,l}$-tracks and $(\Pi'_{m,i,j,k,l})_{m\in\mathbb{N}}$ be a system of pairwise disjoint $\Psi_{k,l} \to \Psi_{i,j}$-tracks. Then

$$\Xi = \left(\bigcup_{i\in\mathbb{N}}\bigcup_{j=1}^{i}\Psi_{i,j}\right) \cup \left(\bigcup_{i\in\mathbb{N}}\bigcup_{j=1}^{i}\bigcup_{k\in\mathbb{N}}\bigcup_{l=1,(i,j)\neq(l,k)}^{k}\bigcup_{m\in\mathbb{N}}(\Pi_{m,i,j,k,l} \cup \Pi'_{m,i,j,k,l})\right)$$

is a connected subdigraph of Δ such that, for $S \sqsubseteq V(\Xi)$, $\Xi - S$ has only one component containing ω-tracks, and every vertex of Ξ lies on an ω-track. Further, for all $n \in \mathbb{N}$, $\mathcal{T}_\Xi(\omega;n) \neq \emptyset$. This shows 2. But there is a difference to the undirected case: Other components of $\Delta - S$ can also be infinite, as Δ may have ω^*-ends or contain infinite zigzag-tracks!

3. Let $\sigma = \{\Theta_1,...,\Theta_n\} \in \mathcal{T}_\Delta(\omega;n)$. Starting from the initial vertex the vertices of Θ_i are denoted by $t_{i1}, t_{i2}, t_{i3}, ...$ $(i \in [n])$. $S \subseteq V(\Delta)$ is called an S_σ, if, for all $i \in [n]$, there exists $k_i \in \mathbb{N}$ such that none of the t_{j1} and none of the t_{jk}, $k \geq k_j$, is contained in S, and any zigzag-track (!) which connects in Δ one of the t_{j1} with a t_{jk}, $k \geq k_j$, contains at least one vertex $\in S$. Clearly, always $|S_\sigma| \geq n$.

4. For all σ there are infinitely many pairwise disjoint finite S_σ.

Can be shown as in [32]. Note, that the fact that Δ can be assumed to be locally finite has to be used here.

5. Let σ be as in 3., for $i \in [n]$ Θ'_i a rest of Θ_i, and $\sigma' = \{\Theta'_1,...,\Theta'_n\}$. Any $S_{\sigma'}$, $|S_{\sigma'}| = n$, is also an S_σ, since if there would be a zigzag-track in Δ between one of the t_{i1} and a vertex w on one of the Θ_i behind $S_{\sigma'}$ there would also be a zigzag-track between one of the t'_{i1} and w,

and S would not be an $S_{\sigma'}$, a contradiction.

6. For any $\sigma = \{\Theta_1, ..., \Theta_n\}$ there is a $\sigma' = \{\Theta'_1, ..., \Theta'_n\}$, where Θ'_i is a rest of Θ_i ($i \in [n]$), such that any $S_{\sigma'}$ contains at least $n+1$ vertices. In addition, σ' can be chosen such that the initials of the Θ'_i either coincide with the initials of the Θ_i, $i \in [n]$, or constitute an S_σ, $|S_\sigma| = n$.

Can be shown as in [32]. The W_{ij} in HALINs proof have here to be chosen as zigzag-tracks Υ_{ij}. Whenever HALIN uses the fact that $G - T$ has exactly one infinite component here the fact that $\Delta - S$ has exactly one component containing ω-tracks has to be used.

7. If S is an S_σ of minimum cardinality (for $\sigma = \{\Theta_1, ..., \Theta_n\}$), then there exist $|S|$ many pairwise disjoint ω-tracks in Δ which originate at the vertices of S. (Of course, these ω-tracks have no vertex in common with the set of vertices which come on Θ_i before $S \cap \Theta_i$.)

Can be shown as in [32]. It is only necessary to apply Lemma 2.16 instead of Satz 3 of [28], to replace "finite component" by "ω-trackless component", and to take into account that every vertex of Δ lies on an ω-track (see 2.). Here, statement 1. is used again.

8. Now, with the help of 6. and 7., it is possible to complete the proof very similar to the proof in [32]. □

2.3 End Orders

This section is dedicated to the following

Problem 3.
Which types of abstract orders can arise on sets of ends of digraphs? □

In other words, which types of abstract orders are ω- (ω*-, c-) representable, respectively? (An abstract poset $(X, \leq)$ is called *ω-representable* if there is a digraph Δ such that the ω-end poset of Δ is order isomorphic to $(X, \leq)$. The notions ω*- and *c-representability* are defined in the same manner.) The following result gives a necessary condition for an order to be ω-representable.

Theorem 2.18.
Let Δ be a digraph and $(\mathcal{E}_i)_{i\in\mathbb{N}}$ be a strictly increasing sequence in $(\Omega(\omega), \leq)$. Then there exists a thick ω-end $\mathcal{F}$ which is the supremum of the sequence $(\mathcal{E}_i)_{i\in\mathbb{N}}$ in $(\Omega(\omega), \leq)$. Moreover, Δ contains a subdivision of the digraph in Fig. 2.3 as subdigraph (compare with HALIN*'s wall graph in Fig. 4 and Satz 4 of [32]).* □

Remark 2.19.
A dual result can be formulated for ω*-tracks by simply substituting "ω" by "ω*", "strictly increasing" by "strictly decreasing", and "supremum" by "infimum", and inverting the orientation of each arc in Fig. 2.3. □

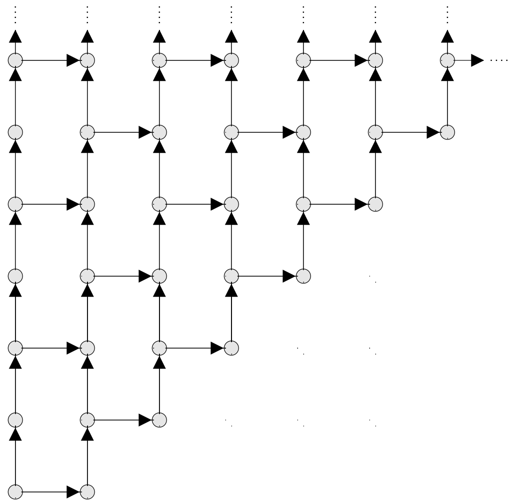


Figure 2.3: see Theorem 2.18

Proof:
Let $(\mathcal{E}_i)_{i\in\mathbb{N}}$ be a strictly increasing sequence in $(\Omega(\omega),\leq)$. Choose $\Theta_1 \in \mathcal{E}_1$ arbitrarily. If, for every $i \in [n]$, $\Theta_i \in \mathcal{E}_i$ is already chosen such that the Θ_i ($i \in [n]$) are pairwise disjoint, then choose $\Theta'_{n+1} \in \mathcal{E}_{n+1}$ arbitrarily. Θ'_{n+1} has only finitely many vertices in common with $\bigcup_{i\in[n]}\Theta_i$. Hence Θ'_{n+1} contains a rest Θ_{n+1} which has no vertex in common with $\bigcup_{i\in[n]}\Theta_i$. By induction, an infinite system of pairwise disjoint ω-tracks $(\Theta_i)_{i\in\mathbb{N}}$ with $\Theta_i \in \mathcal{E}_i$ is obtained. Let the vertices of Θ_i — beginning with the initial vertex — be denoted by $u_{i,1}$, $u_{i,2}$, $u_{i,3},\ldots$ ($i \in \mathbb{N}$). For every $i \in \mathbb{N}$ and $n \in \mathbb{N}$, let $\Theta_i(n)$ denote the subtrack of Θ_i with the vertices $u_{i,1}$, $u_{i,2}$,..., $u_{i,n}$. The following two statements will be needed in the sequel:

(a) For $i, j \in \mathbb{N}$, $i < j$, there exist infinitely many pairwise disjoint $\Theta_i \to \Theta_j$–tracks.

(b) For all $i \in \mathbb{N}$ and all $M \sqsubseteq \mathbb{N}$ with $i, i+1 \notin M$, there exist infinitely many pairwise disjoint $\Theta_i \to \Theta_{i+1}$–tracks each not meeting Θ_k for all $k \in M$.

    (Otherwise, there exist infinitely many pairwise disjoint $\Theta_i \to \Theta_{i+1}$–tracks such that each of them intersects with at least one of the Θ_k, $k \in M$. Since M is finite, there exists $k' \in M$ such that infinitely many pairwise disjoint $\Theta_i \to \Theta_{i+1}$–tracks intersect with $\Theta_{k'}$. If $k' < i$, then $\Theta_{k'} \geq \Theta_i$, and hence $\mathcal{E}_{k'} \geq \mathcal{E}_i$, a contradiction. If $k' > i+1$, then $\Theta_{i+1} \geq \Theta_{k'}$, and hence $\mathcal{E}_{i+1} \geq \mathcal{E}_{k'}$, a contradiction as well.)

An overview of the remaining proof can be given as follows: By induction, infinitely many pairwise disjoint ω-tracks Ψ_i ($i \in \mathbb{N}$) are constructed which belong to an ω-end $\mathcal{F}$, which turns out to be the supremum of $(\mathcal{E}_i)_{i\in\mathbb{N}}$ and whose union with the $(\Theta_i)_{i\in\mathbb{N}}$ contains a subdivision of the digraph in Fig. 2.3.

Clearly, there exists a $\Theta_1 \to \Theta_2$–track $\Psi_{1,1}$. For the purpose of cutting away everything that lies on a Θ_i "under" $\Psi_{1,1}$, let

$$\Xi_1 := \Psi_{1,1} \cup \bigcup\{\Theta_i(n) \mid \exists i,n \in \mathbb{N} : u_{i,n} \in \Psi_{1,1} \cap \Theta_i\}.$$

Ξ_1 is finite.

Because of (a), there exists a $\Theta_1 \to \Theta_2$–track $\Pi^1_{1,2}$ in $\Delta - \Xi_1$. Because of (b), there exists a $\Theta_2 \to \Theta_3$–track $\Psi_{1,2}$ and a $\Theta_1 \to \Theta_2$–track $\Psi_{2,1}$ such that $\Psi_{1,2}$ does not meet Θ_1, the initial vertex of $\Psi_{1,2}$ comes behind the terminal vertex of $\Pi^1_{1,2}$ on Θ_2, $\Psi_{2,1}$ does not meet Θ_3 and $\Psi_{1,2}$, and the terminal vertex of $\Psi_{2,1}$ comes behind the initial vertex of $\Psi_{1,2}$ on Θ_2. For the purpose of cutting away everything that lies on a Θ_i "under" $\Psi_{1,2}$, $\Psi_{2,1}$, and $\Pi^1_{1,2}$, let

$$\Upsilon_2 := \Psi_{1,2} \cup \Psi_{2,1} \cup \Pi^1_{1,2}$$

and

$$\Xi_2 := \Xi_1 \cup \Upsilon_2 \cup \bigcup \{\Theta_i(n) \mid \exists i,n \in \mathbb{N} : u_{i,n} \in \Upsilon_2 \cap \Theta_i\}.$$

Ξ_2 is finite.

Now, let $\Psi_{1,n}$, $\Psi_{2,n-1}$,..., $\Psi_{n,1}$ and $\Pi^k_{i,n-k+1}$ $(k \in [n-1], i < n-k+1)$ and Ξ_n already be constructed. Because of (a), there exist $\Theta_i \to \Theta_{n-k+2}$–tracks $\Pi^k_{i,n-k+2}$ $(k \in [n], i < n-k+2)$ in $\Delta - \Xi_n$. Because of (b), there exists a $\Theta_{n+1} \to \Theta_{n+2}$–track $\Psi_{1,n+1}$, a $\Theta_n \to \Theta_{n+1}$–track $\Psi_{2,n}$,..., and a $\Theta_1 \to \Theta_2$-track $\Psi_{n+1,1}$ such that $\Psi_{1,n+1}$ does not meet Θ_1, Θ_2,..., Θ_n, the initial vertex of $\Psi_{1,n+1}$ comes behind the terminal vertices of the $\Pi^1_{i,n+1}$ on Θ_{n+1} $(i \in [n])$, $\Psi_{2,n}$ does not meet Θ_1,..., Θ_{n-1}, Θ_{n+2}, and $\Psi_{1,n+1}$, the initial vertex of $\Psi_{2,n}$ comes behind the terminal vertices of the $\Pi^2_{i,n}$ on Θ_n $(i \in [n-1])$, the terminal vertex of $\Psi_{2,n}$ comes behind the initial vertex of $\Psi_{1,n+1}$ on Θ_{n+1},..., $\Psi_{n+1,1}$ does not meet Θ_3, Θ_4,..., Θ_{n+2}, $\Psi_{1,n+1}$,..., $\Psi_{n-1,3}$, and $\Psi_{n,2}$, and the terminal vertex of $\Psi_{n+1,1}$ comes behind the initial vertex of $\Psi_{n,2}$ on Θ_2. Let

$$\Upsilon_{n+1} := \left(\bigcup_{i \in [n+1]} \Psi_{i,n+2-i} \right) \cup \left(\bigcup_{k \in [n], i \in [n-k+1]} \Pi^k_{i,n-k+2} \right)$$

and

$$\Xi_{n+1} := \Xi_n \cup \Upsilon_{n+1} \cup \bigcup \{\Theta_i(n) \mid \exists i,n \in \mathbb{N} : u_{i,n} \in \Upsilon_{n+1} \cap \Theta_i\}.$$

Ξ_{n+1} is finite.

After completing the induction, for $i, j \in \mathbb{N}$, let $\Psi'_{i,j}$ be the union of $\Psi_{i,j}$ with the segment of Θ_{j+1} between the terminal vertex of $\Psi_{i,j}$ and the initial vertex of $\Psi_{i,j+1}$ and set

$$\Psi_i := \bigcup_{j \in \mathbb{N}} \Psi'_{i,j} \quad (i \in \mathbb{N}).$$

By construction, the Ψ_i $(i \in \mathbb{N})$ are pairwise disjoint, and, clearly, the union of the Ψ_i with the Θ_i $(i \in \mathbb{N})$ contains a subdivision of the digraph in Fig. 2.3.

Ψ_1 belongs to an ω-end $\mathcal{F}$ and is constructed such that, for each $i \in \mathbb{N}$, there are infinitely many $\Theta_i \to \Psi_1$-tracks $\Pi^1_{i,j}$ $(j \in \mathbb{N} \setminus \{1\})$. Hence, for all $i \in \mathbb{N}$, $\Theta_i \leq \Psi_1$, and, therefore, for all $i \in \mathbb{N}$, $\mathcal{E}_i \leq \mathcal{F}$. $\mathcal{F} = \mathcal{E}_i$ for an $i \in \mathbb{N}$ is impossible because $\mathcal{F} = \mathcal{E}_i < \mathcal{E}_{i+1} \leq \mathcal{F}$ yields a contradiction. Hence, $\mathcal{F} > \mathcal{E}_i$ for all $i \in \mathbb{N}$.

Now, let $\mathcal{E} \in \Omega(\omega)$ with $\mathcal{E} > \mathcal{E}_i$ for all $i \in \mathbb{N}$. To show that $\mathcal{F}$ is the supremum of the sequence $(\mathcal{E}_i)_{i \in \mathbb{N}}$, it remains to show that $\mathcal{E} \geq \mathcal{F}$. $\mathcal{E}$ contains an ω-track Ψ. Since $\mathcal{E} > \mathcal{E}_1$ and $\Psi_1 \cap \Theta_1$ is finite, it is possible to find a $\Theta_1 \to \Psi$–track Λ'_1 such that the initial vertex of Λ'_1 comes behind $\Psi_1 \cap \Theta_1$ on Θ_1. Obviously, the union of Λ'_1 with the segment of Θ_1 between the last vertex of $\Psi_1 \cap \Theta_1$ and the initial vertex of Λ'_1 contains a $\Psi_1 \to \Psi$–track Λ_1. Let $i_1 := 1$.

For $n \in \mathbb{N}$, let the pairwise disjoint $\Psi_1 \to \Psi$–tracks $\Lambda_1, ..., \Lambda_n$ already be constructed. Since the Λ_i ($i \in [n]$) intersect only with finitely many of the Θ_i ($i \in \mathbb{N}$) in Δ, there exists $i_{n+1} \in \mathbb{N}$ such that $\Theta_{i_{n+1}}$ does not intersect with one of the Λ_i ($i \in [n]$). Because Ψ_1 intersects with all Θ_i ($i \in \mathbb{N}$) and because $\mathcal{E} > \mathcal{E}_i$ ($i \in \mathbb{N}$), it is possible to find a $\Psi_1 \to \Psi$–track Λ_{n+1} in $\Delta - (\bigcup_{i \in [n]} \Lambda_i)$, just as in the case $n = 1$.

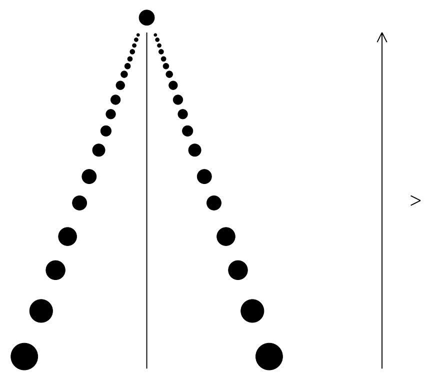
By induction, one obtains a system of infinitely many pairwise disjoint $\Psi_1 \to \Psi$–tracks $(\Lambda_i)_{i \in \mathbb{N}}$. Hence, $\mathcal{E} \geq \mathcal{F}$.

It remains to show that $\mathcal{F}$ is thick. Repeating the argumentation of the last four paragraphs for each Ψ_i ($i \geq 2$) yields $\Psi_i \in \mathcal{F}$ for all $i \in \mathbb{N}$ since suprema are uniquely determined. As the Ψ_i are pairwise disjoint, $\mathcal{F}$ is a thick ω-end, and the proof is completed. □

The next theorem implies that the necessary condition of Theorem 2.18 is also sufficient in the class of total orders. Naturally, a poset $(X, \leq)$ can be regarded as a reflexive, antisymmetric, and transitive digraph $\Delta = (X, \leq)$ (and vice versa). There exists a natural bijective mapping between the set of all strictly increasing sequences of $(X, \leq)$ and $\mathcal{T}_\Delta(\omega)$. Now, if $(X, \leq)$ is a chain fulfilling the condition that all strictly increasing sequences have a supremum, then

(*) two ω-tracks Θ, Ψ belong to the same ω-end if and only if the corresponding strictly increasing sequences $(u_i)_{i \in \mathbb{N}}$ and $(v_i)_{i \in \mathbb{N}}$ have the same supremum $u = v$.

(If $(u_i)_{i \in \mathbb{N}}$ and $(v_i)_{i \in \mathbb{N}}$ did not have the same supremum, then $u < v$ or $v < u$ since $(X, \leq)$ is a chain. Without loss of generality, assume $u < v$. Then there would exist $n \in \mathbb{N}$ such that $v_j > u_i$ for all $i \in \mathbb{N}$ and $j \geq n$. Thus, no u_i ($i \in \mathbb{N}$) would be reachable by a v_j, $j \geq n$. Hence, $\mathcal{E}_\Delta(\omega, \Theta) < \mathcal{E}_\Delta(\omega, \Psi)$, a contradiction. The other direction is trivial.)



(*) is not necessarily true for general posets. For example, Fig. 2.4 shows a poset consisting of two completely independent, strictly increasing sequences with a common supremum. From statement (*), it remains only: Θ, Ψ belong to the same ω-end *only if* the corresponding strictly increasing sequences have the same supremum.

Figure 2.4: The runway poset.

The validity of (*) is very convenient for constructing digraphs whose ω-end order represents $(X, \leq)$. Just let each $x \in X$ that occurs as the supremum of a strictly increasing sequence be represented by that ω-end which contains the ω-tracks whose corresponding strictly increasing sequences have x as their supremum. If (*) is not fulfilled, it seems to be very difficult to find a digraph whose ω-end structure represents $(X, \leq)$ — and, therefore, (*) is postulated in Theorem 2.20. For further investigations, the following remarks may be useful:

Let M be the set of all $x \in X$ that are not the supremum of any strictly increasing sequence in X. For $x \in X \setminus M$, let σ_x denote the (nonempty) set of all strictly increasing sequences of X that have the supremum x. Obviously, σ_x decomposes into a family of equivalence classes $(\mathcal{S}_i^x)_{i \in I_x}$ such that the corresponding classes of ω-tracks are subsets of ω-ends $(\mathcal{E}_i)_{i \in I_x}$ of $(X, \leq)$, which is partially ordered corresponding to the ω-end-subordering of the $(\mathcal{E}_i)_{i \in I_x}$. One may be tempted to add arcs to weld the ω-ends $(\mathcal{E}_i)_{i \in I_x}$ together, but then at least two problems can occur:

1. Adding new arcs may generate new ω-ends that have nothing to do with $(X, \leq)$.
2. Too many ω-ends may possibly be welded together if new arcs are added globally for all suprema.

The third condition of Theorem 2.20 is required to preserve the order structure of $(X, \leq)$ at suprema for the ω-end-structure of the digraph constructed in the proof of Theorem 2.20.

Theorem 2.20.
Let $(X, \leq)$ be a poset. It is ω-representable if the following conditions are fulfilled:

(1) Every strictly increasing sequence in $(X, \leq)$ has a supremum in X.

(2) Whenever two strictly increasing sequences have the same supremum, the corresponding ω-tracks of $(X, \leq)$ belong to the same ω-end of $(X, \leq)$.

(3) If $x \in X$ is the supremum of a strictly increasing sequence in $(X, \leq)$, then, for all $y \in X$, $y < x$, there exists a strictly increasing sequence $(z_i^{y,x})_{i \in \mathbb{N}}$ with supremum $z = x$ and $y < z_i^{y,x} < x$ for all $i \in \mathbb{N}$. □

Proof:
Let M be defined as above. A digraph Δ can be constructed from $(X, \leq)$ as follows: For each $x \in M$, let Ψ_x denote an ω-track with the vertex set $\{x, x_i \mid i \in \mathbb{N}\}$ such that, for all $x, y \in M$, Ψ_x and Ψ_y are disjoint whenever $x \neq y$, and, for all $x \in M$, $\Psi_x \cap (X, \leq) = x$. Now, let

$$A' := \{(z, x_i), (x_i, z'), (x_i, y_j) \mid x, y \in M, z, z' \in X, z < x < z', x < y, i, j \in \mathbb{N}\}$$

and

$$\Delta := (X, \leq) \cup \bigcup_{x \in M} \Psi_x \cup A'.$$

Clearly, $x \leq y \Leftrightarrow x \rightarrow_\Delta y$ for all $x, y \in X$.

(A) Each $\Theta \in \mathcal{T}_\Delta(\omega)$ has either a rest in common with exactly one of the Ψ_x or only finitely many vertices in common with each of the Ψ_x ($x \in M$). (If a track leaves Ψ_x it can never return to Ψ_x.) In the latter case, Θ corresponds in a natural way to a strictly increasing sequence of $(X, \leq)$.

(B) $\Theta, \Theta' \in \mathcal{T}_\Delta(\omega)$ belong to the same ω-end of Δ if and only if either rests of Θ and Θ' belong to the same Ψ_x or both Θ and Θ' correspond to strictly increasing sequences $(u_i)_{i \in \mathbb{N}}$, $(v_i)_{i \in \mathbb{N}}$ in $(X, \leq)$ that have the same supremum $u = v \in X$.

$\Leftarrow$: Trivial if rests of Θ and Θ' belong to the same Ψ_x. If Θ and Θ' correspond to strictly increasing sequences in $(X, \leq)$ with the same supremum, then '$\Leftarrow$' follows from (2) and the fact that all $\Psi_x \rightarrow \Psi_y$-arcs exist whenever $x < y$.

$\Rightarrow$: Because of (A) and (1), only three cases have to be considered:

(I) A rest of Θ belongs to Ψ_x and a rest of Θ' belongs to Ψ_y.

(II) A rest of Θ belongs to Ψ_x and Θ' corresponds to a strictly increasing sequence $(v_i)_{i\in\mathbb{N}}$ with supremum v in $(X,\leq)$.

(III) Both Θ and Θ' correspond to a strictly increasing sequence (with suprema u and v, respectively) in $(X,\leq)$.

ad (I): If $x \neq y$, then (without loss of generality) either $x < y$ or x and y are incomparable. In the former case, $\Theta < \Theta'$ is valid, implying that $\mathcal{E}_\Delta(\omega,\Theta) < \mathcal{E}_\Delta(\omega,\Theta')$, and, in the latter case, Θ and Θ' are incomparable, implying that $\mathcal{E}_\Delta(\omega,\Theta)$ and $\mathcal{E}_\Delta(\omega,\Theta')$ are incomparable. In both cases, Θ and Θ' do not belong to the same ω-end of Δ, a contradiction. Hence $x = y$.

ad (II): There are three possible cases ($v = x$ is impossible since $x \in M$, $v \notin M$):

(a) $v < x$: Then $v_i < x$ for all $i \in \mathbb{N}$, and hence $\Theta' < \Theta$ (since there is no $\Theta \to \Theta'$–track in Δ and, for every $i \in \mathbb{N}$, there exist the arc (v_i,x_i)), implying that $\mathcal{E}_\Delta(\omega,\Theta') < \mathcal{E}_\Delta(\omega,\Theta)$. Therefore, Θ and Θ' do not belong to the same ω-end, a contradiction.

(b) $v > x$: Then, because of (3), there exists a strictly increasing sequence $(z_i^{x,v})_{i\in\mathbb{N}}$ with supremum $z = v$. Because of (2), the ω-track Π with vertices $(z_i^{x,v})_{i\in\mathbb{N}}$ is end-equivalent to the ω-track Ψ' with vertices $(v_i)_{i\in\mathbb{N}}$ (and clearly, $\Psi' \sim \Theta'$). Hence, $\Pi \sim \Theta'$. Because $(x_i,z_i) \in A(\Delta)$ for all $i \in \mathbb{N}$ and there exists no $\Pi \to \Theta$–track in Δ, $\Theta < \Pi \sim \Theta'$, and hence $\mathcal{E}_\Delta(\omega,\Theta) < \mathcal{E}_\Delta(\omega,\Theta')$, a contradiction.

(c) v and x are incomparable: $v_i < x$ for all $i \in \mathbb{N}$ is not possible because, otherwise, $v \leq x$ since v is the supremum of $(v_i)_{i\in\mathbb{N}}$. If there exists $j \in \mathbb{N}$ such that $v_j \geq x$, then $v > v_j \geq x$, which is impossible, too. Hence, there exists $n \in \mathbb{N}$ such that v_i and x are incomparable for all $i \geq n$. It follows that there are no $\Theta \to \Theta'$–tracks and only finitely many pairwise disjoint $\Theta' \to \Theta$–tracks in Δ. Hence Θ and Θ', and therefore $\mathcal{E}_\Delta(\omega,\Theta)$ and $\mathcal{E}_\Delta(\omega,\Theta')$, are incomparable, a contradiction.

Altogether, case (II) is impossible.

ad (III): Again, there are three possible cases:

(a) $u = v$: finished.

(b) Without loss of generality, $u < v$ ($v < u$ is a symmetric situation): It is possible to argue as in (II)(b) by substituting "x_i" with "u_i" and "x" with "u". In particular, $\mathcal{E}_\Delta(\omega,\Theta) < \mathcal{E}_\Delta(\omega,\Theta')$, a contradiction.

(c) u and v are incomparable: $u_i < v$ for all $i \in \mathbb{N}$ is not possible because, otherwise, $u \leq v$ since u is the supremum of $(u_i)_{i\in\mathbb{N}}$. Symmetrically, $v_i < u$ for all $i \in \mathbb{N}$ is impossible. If there exists $j \in \mathbb{N}$ such that $u_j > v$, then $u > u_j > v$, which is impossible, too. Symmetrically, $v_j > u$ for a $j \in \mathbb{N}$ is impossible. It follows, that there exist $m,n \in \mathbb{N}$ such that u_i and v_j are incomparable for all $i \geq m$, $j \geq n$. So if Λ denotes the ω-track with the vertices $(u_i)_{i\in\mathbb{N}}$ and Λ' the ω-track with the vertices $(v_i)_{i\in\mathbb{N}}$, then Λ and Λ' are incomparable in $(\mathcal{T}_\Delta(\omega),\leq)$. Since $\Theta \sim \Lambda$ and $\Theta' \sim \Lambda'$, Θ and Θ', and therefore $\mathcal{E}_\Delta(\omega,\Theta)$ and $\mathcal{E}_\Delta(\omega,\Theta')$, are incomparable, a contradiction. This completes the proof of (B).

Now, consider the mapping $f : X \to \Omega_\Delta(\omega)$, $x \mapsto \mathcal{E}_x$, where, for $x \in M$ ($x \notin M$), $\mathcal{E}_x$ denotes the ω-end of Δ corresponding to Ψ_x (the ω-end whose elements correspond to strictly increasing sequences in $(X, \leq)$ with the supremum x), respectively.

f is well defined: Because of (B), each Ψ_x ($x \in M$) represents exactly one ω-end of Δ. If $x \notin M$, then x is the supremum of a strictly increasing sequence S in $(X, \leq)$. There is at least one ω-track of Δ that corresponds to S. This, in combination with (B), guarantees the existence of exactly one ω-end fulfilling the definition of $\mathcal{E}_x$ in the case $x \notin M$.

Because of (A) and (1), f is surjective. That f is injective and is an order isomorphism follows from the proof of (B). □

Conditions (2) and (3) of Theorem 2.20 are not necessary: There are posets not fulfilling (2) or (3) which are nevertheless ω-representable. For example, the runway-poset (Fig. 2.4) is ω-represented by the digraph Δ_1 in Fig. 2.2, but, obviously, it violates (2). Now, let $(X, \leq)$ be a poset where $X = \{x, y, x_i \mid i \in \mathbb{N}\}$, $(x_i)_{i \in \mathbb{N}}$ be a strictly increasing sequence with the supremum x, and y be an element less than x but independent from each x_i, $i \in \mathbb{N}$. Now, let Δ be constructed as in Theorem 2.20 and include the following additional arcs: (y_i, x_i) for all $i \in \mathbb{N}$. It is easy to see that $(X, \leq)$ is ω-represented by Δ in spite of the fact that $(X, \leq)$ does not fulfill condition (3).

Condition (1) of Theorem 2.20 characterizes those total orders that are ω-representable, since conditions (2) and (3) are always fulfilled in chains. It is possible to obtain a dual result for ω^*-tracks (see Remark 2.19).

Corollary 2.21.
Let $(X, \leq)$ be a chain. Then $(X, \leq)$ is ω-representable if and only if every strictly increasing sequence of $(X, \leq)$ has a supremum. □

The conditions (1), (2), and (3) are always fulfilled in posets that contain no strictly increasing sequence, which are exactly the posets in which every chain is a dwoset. Therefore:

Corollary 2.22.
Every poset that contains only dually well-ordered chains is ω-representable. □

Up to this point, only ω- and ω^*-representability have been studied. It remains to investigate c-representability. Whether an abstract order is c-representable or not seems to be a completely different problem:

- There are orders that are c- but neither ω- nor ω^*-representable, e.g. $\mathbb{Z}$ with the natural order (see Example 2.28).
- An ω-representable order $(X, \leq)$ may not be c-representable because each digraph whose ω-end order represents $(X, \leq)$ may also contain ω^*-tracks that are contained in a c-end which contains no ω-tracks.

Up to now, I was not able to find an example to illustrate the second of the above points. Therefore, the following problem may be interesting.

Problem 4.
Are there abstract orders that are ω- but not c-representable? □

As composed ends contain both ω- and ω^*-tracks, it is a little bit more difficult to find a necessary condition in terms of infima/suprema for an abstract order to be c-representable. One has to draw on a more special and sophisticated looking class of orders. First of all, the following lemma is needed:

Lemma 2.23.
Let C be an (inclusion-)maximal chain in $(\Omega_\Delta(c), \leq)$ and $\Omega_\Delta^\omega(c)$ be the set of all c-ends of Δ that contain exclusively ω-tracks. If C contains a countable, upper unbounded subset, then $\Omega_\Delta^\omega(c) \cap C$ is upper bounded in C. A dual statement holds for ω^*-tracks. □

Proof:
Let $A = \{a_i \mid i \in \mathbb{N}\}$ be an upper unbounded, countable subset of C and $C' := \Omega_\Delta^\omega(c) \cap C$. If C' is upper unbounded in C, then, for every $i \in \mathbb{N}$, there exists a $c_i \in C'$ such that $c_i > a_i$, i.e. $\{c_i \mid i \in \mathbb{N}\}$ is upper unbounded. Clearly, $\{c_i \mid i \in \mathbb{N}\}$ contains an upper unbounded strictly increasing sequence, and Theorem 2.18 yields a contradiction to the maximality of C. □

The countability of the unbounded subset of C is necessary because there exist orders in which every countable subset is bounded, e.g. any order of order type ω_1 (see [1], pp.65–67, Theorems 17, 17', 17").

The following theorem gives a sufficient condition for an order not to be c-representable. (A,B) is called a *cut of* $(X, \leq)$ if $A \cap B = \emptyset, A \cup B = X$, and $a < b$ holds for all $a \in A, b \in B$. In addition, if A and B are nonempty, then (A,B) is called a *proper cut of* $(X, \leq)$.

Theorem 2.24.
Let $(X, \leq)$ be a poset and C be an (inclusion-)maximal chain in $(X, \leq)$ containing an upper unbounded, countable subset. Furthermore, let $M = M(C)$ be the nonempty set of all proper cuts (A,B) of C such that A has no greatest element and B contains a lower unbounded countable subset, and let $\bigcap_{(A,B) \in M} B = \emptyset$. Then $(X, \leq)$ is not c-representable. A dual statement holds if C contains a lower unbounded, countable subset by changing the roles of A and B.

Proof:
Assume C to be a maximal chain in $(X, \leq)$ such that C contains an upper unbounded, countable subset and $\bigcap_{(A,B) \in M} B = \emptyset$. Let Δ be a digraph and $f : X \to \Omega_\Delta(c)$ be an (order) isomorphism. Then $f[C]$ is a maximal chain in $(\Omega_\Delta(c), \leq)$ and contains an upper unbounded, countable subset. From Lemma 2.23 it follows that $\Omega_\Delta^\omega(c) \cap f[C]$ is upper bounded. Let $f(x)$ be an upper bound of $\Omega_\Delta^\omega(c) \cap f[C]$ in $f[C]$. Since $\bigcap_{(A,B) \in M} B = \emptyset$ and M is nonempty, there exists a cut $(A', B') \in M$ such that $x \notin B'$, i.e. $f(x) \notin f[B']$. It follows that $f(x) < \mathcal{E}$ for all $\mathcal{E} \in f[B']$. B' contains a lower unbounded strictly decreasing sequence $(x_i)_{i \in \mathbb{N}}$. According to Theorem 2.18, there exists $\mathcal{F} \in f[C]$ such that $\mathcal{F}$ is the infimum of the sequence $(f(x_i))_{i \in \mathbb{N}}$. Clearly, $\mathcal{F} \in A'$ since $(f(x_i))_{i \in \mathbb{N}}$ is lower unbounded in $f[B']$, which is a contradiction because A' does not contain a greatest element. □

Example 2.25.
$\mathbb{Q}$ with the natural order is not c-representable since $\mathbb{Q}$ is a countable, infinite chain, and the set of all gaps, which are characterizable with the help of cuts, is both upper and lower unbounded. Clearly, $\mathbb{Q}$ with the natural order is neither ω- nor ω^*-representable, and hence there are orders that are not at all end-representable by digraphs. □

The next two theorems will show that there are orders that are c- but not ω- (resp. ω^*-) representable.

Theorem 2.26.
Every poset in which every chain is well ordered (resp. dually well ordered) is c-representable.

Proof:
Let $(X,\leq)$ be a poset such that every chain in $(X,\leq)$ is dually well ordered. Then $(X,\leq)$ contains no strictly increasing sequence. Construct a digraph Δ as follows: First of all, for each $x\in X$, let Δ contain an ω-track Ψ_x such that the Ψ_x $(x\in X)$ are pairwise disjoint and the vertices of Ψ_x — beginning with the initial vertex — are denoted by $w_{x,1}$, $w_{x,2}$, $w_{x,3},\ldots$ Additionally, let Δ contain the arcs $(w_{x,n},w_{y,n+1})$ $(n\in\mathbb{N},x,y\in X,x<y)$. With that, the construction of Δ is completed.

Clearly, each Ψ_x $(x\in X)$ represents another c-end of Δ. Further, $x\leq y$ if and only if $\mathcal{E}_\Delta(c,\Psi_x)\leq\mathcal{E}_\Delta(c,\Psi_y)$ for all $x,y\in X$ because, for $x<y$, there are infinitely many $\Psi_x\to\Psi_y$–arcs in Δ, but no $\Psi_y\to\Psi_x$–tracks. Hence, $(\Omega_\Delta(c),\leq)\simeq(X,\leq)$ provided that Δ contains no other c-ends except the $\mathcal{E}_\Delta(c,\Psi_x)$, $x\in X$.

Let $\Theta\in\mathcal{T}_\Delta(\omega)$. Then either Θ has a rest in common with one of the Ψ_x or Θ corresponds to a strictly increasing sequence of $(X,\leq)$. Since $(X,\leq)$ contains no strictly increasing sequence, Θ must belong to one of the $\mathcal{E}_\Delta(c,\Psi_x)$. Now, let $\Psi\in\mathcal{T}_\Delta(\omega^*)$. Ψ contains a vertex $w_{x,n}$. By the construction of Δ, there are vertices $w_{y,n-1}$,..., $w_{z,1}$ on Ψ, but no arc terminates at $w_{z,1}$, a contradiction. Hence, Δ does not contain ω^*-tracks, and the theorem is proved. □

Let $(X,\leq)$ and $(Y_x,\leq_x)$ $(x\in X)$ be posets. Then $\sum_{x\in X}Y_x$, the $(X,\leq)$-*sum of the* $((Y_x,\leq))_{x\in X}$, denotes the poset

$$(\{(x,y)\mid x\in X,y\in Y_x\},\{((x,y),(x',y'))\mid \text{either } x=x' \text{ and } y\leq_x y' \text{ or } x<x'\}).$$

Theorem 2.27.
Let $(X,\leq)$ be a poset containing only finite chains and $(Y_x)_{x\in X}$ be a family of c-representable posets. Then $\sum_{x\in X}Y_x$ is c-representable.

Proof:
Let $(D_x)_{x\in X}$ be a family of pairwise disjoint digraphs such that Δ_x c-represents Y_x $(x\in X)$. If one welds together the $(D_x)_{x\in X}$ in the same manner as the $(Y_x)_{x\in X}$, then it is easy to see that the resulting digraph Δ c-represents $\sum_{x\in X}Y_x$: Since $(X,\leq)$ contains only finite chains, each 1-track in Δ has a rest in common with a 1-track of one of the Δ_x $(x\in X)$. □

A version of Theorem 2.27 for ω-representable posets is also valid. In this version $(X,\leq)$ can be allowed to be a poset containing only dually well ordered (resp. well ordered) chains.

Example 2.28.
$\mathbb{Z}$ with the natural order is c-representable (by combination of Theorem 2.26 and Theorem 2.27). □

This section does not provide a full characterization of those orders that are end-representable by arbitrary digraphs. Maybe this is possible in smaller classes of digraphs. For example, the following problems may be interesting:

Problem 5.
Which abstract orders are end-representable by locally finite digraphs? □

Problem 6.
Which abstract orders are end-representable by vertex-transitive digraphs? □

Further, it can be expected that there are orders which are neither ω-representable nor fulfilling condition (2) or (3) of Theorem 2.20, but in which every strictly increasing sequence has a supremum. A suitable counterexample would increase the expectation that a simple characterization of ω-representable orders cannot be found.

2.4 Endfaithful Sub- and Superdigraphs

Given a digraph Δ, it is interesting to find simpler super- and subdigraphs of Δ that have the same end structure as Δ. In this section, characterizations of those digraphs are given which are endfaithful with Δ^s and Δ^t, respectively. Further, a result of HALIN about endfaithful spanning trees is generalized to digraphs. For the purpose of a simpler notation, 1-tracks may also be called c-tracks throughout this section.

For $i \in \{\omega, \omega^*, c\}$, a subgraph Ξ of a digraph Δ is called *i-endinjective in* Δ if, for every pair Θ, Θ' of i-tracks of Ξ that belong to the same i-end of Δ, Θ and Θ' belong also to the same i-end of Ξ. Ξ is called *i-endcomplete in* Δ if, for every i-end $\mathcal{E}$ of Δ, there exists an i-track Θ of Ξ such that $\Theta \in \mathcal{E}$. Ξ is called *i-endfaithful with* Δ if Ξ is both i-endinjective and i-endcomplete in Δ.

Two ends $\mathcal{E}$, $\mathcal{F}$ of a digraph Δ are said to be *(strongly) separated by an* $F \sqsubset V(\Delta)$ if, for every $\Theta \in \mathcal{E}$ and every $\Psi \in \mathcal{F}$, $rest(\Theta - F)$ and $rest(\Psi - F)$ are separated in $\Delta - F$.

The proof of the next theorem is very simple and can be found in [93] (Satz 20).

Theorem 2.29.
If Δ is a digraph and $i \in \{\omega, \omega^*, c\}$, then the following statements are valid:

(1) Δ is i-endinjective in Δ^s if and only if every pair of i-ends of Δ can be separated by some finite subset of $V(\Delta)$.

(2) Δ is i-endcomplete in Δ^s if and only if, for every one-way infinite zigzag-track Θ of Δ, there is an i-track Ψ such that there are infinitely many pairwise disjoint Θ,Ψ–zigzag-tracks. □

The subsequent lemma is a generalization of the so-called "KÖNIGslemma" (see [51], Kap. VI, Satz 3) to digraphs and can be proved in the same way:

Lemma 2.30.
If Δ is an inner (outer) locally finite digraph then Δ contains an ω-track (ω-track) if and only if Δ contains a vertex with infinite transmitting (receiving) area, respectively.* □

Theorem 2.31.
If Δ is a digraph and $i \in \{\omega, \omega^*, c\}$, then the following statements are valid:

(1) Δ is i-endinjective in Δ^t if and only if every strong component of Δ has at most one i-end.

(2) Δ is i-endcomplete in Δ^t if and only if every infinite strong component of Δ has at least one i-end.

(3) Δ is i-endfaithful with Δ^t if and only if every infinite strong component of Δ has exactly one i-end. □

Proof:
(1), (2), (3) '$\Rightarrow$': If Ξ is a strong component of Δ, then Ξ^t is a complete digraph and has an i-end (exactly one) if and only if it is infinite. Now, look at the definitions of endinjectiveness, endcompleteness, and endfaithfulness.

(1) '$\Leftarrow$': Let Θ and Θ' be 1-tracks of Δ that are contained in different i-ends of Δ. Then one can distinguish three cases.

The first case is that Θ and Θ' are both cyclic. Then subrays of Θ and Θ' belong to different strong components Ξ and Ξ' of Δ, respectively, since strong components contain at most one i-end. Since the reachability relation induces a partial order on the set of strong components of Δ, at most one of the strong components Ξ and Ξ' is reachable by the other. Hence, Θ and Θ' belong to different i-ends of Δ^t.

The second case is that Θ is cyclic and Θ' is acyclic (or vice versa). Since the set of i-ends of Δ is partitioned into cyclic and acyclic ends, Θ and Θ' belong to different ends of Δ^t.

The third case is that Θ and Θ' are both acyclic. Assume that Θ and Θ' belong to the same end of Δ^t. Either there are infinitely many pairwise disjoint $\Theta \to \Theta'$–tracks or Θ' can be weakly separated from Θ in Δ by some $S \sqsubseteq \Delta$. Clearly, in the latter case, there is a vertex $v \in S$ such that v is reachable by infinitely many (and hence almost all) vertices of Θ and infinitely many (and hence almost all) vertices of Θ' are reachable from v because Θ and Θ' belong to the same end of Δ^t. An analogous statement holds by exchanging the roles of Θ and Θ' (with a separating set S' and a vertex v'). Note, that it is not possible that there are infinitely many pairwise disjoint $\Theta' \to \Theta$–tracks as well as infinitely many pairwise disjoint $\Theta \to \Theta'$–tracks in Δ. Now it is easy to see that, in any case, rests of Θ and Θ' belong to the same strong component of Δ which contradicts the fact that Θ is acyclic.

(2) '$\Leftarrow$': Assume that every infinite strong component of Δ contains at least one i-end. If $\mathcal{E}$ is a cyclic end of Δ^t then there is no problem since strong components of Δ become maximum complete subgraphs of Δ^t, and the strong components of Δ^t are exactly its maximum complete subgraphs.

If $\mathcal{E}$ is an acyclic end of Δ^t, then assume that there does not exist a 1-track of Δ which belongs to $\mathcal{E}$ in Δ^t. Let $\Theta \in \mathcal{E}$. Since Δ contains no rest of Θ, there are infinitely many edges of Θ that are not contained in Δ.

Now, consider Θ to be an ω-track, and, by turns, starting from the initial vertex, denote the edges of Θ that are not contained in Δ as (v_i, w_i), $i \in \mathbb{N}$. Since (v_i, w_i) is an edge of Δ^t, there exists a $v_i \to w_i$–track Π_i in Δ. Let

$$\Xi := \left(\bigcup_{i \in \mathbb{N}} \Pi_i \right) \cup \left(\Theta - \bigcup_{i \in \mathbb{N}} (v_i, w_i) \right).$$

Ξ is an infinite, weakly connected subdigraph of Δ. Hence Ξ cannot contain a 1-track ($i = c$) (ω-track ($i = \omega$)) Θ', respectively, since Θ' would meet infinitely many of the Π_i and, therefore, for every $S \sqsubseteq V(\Delta^t)$, the relation $\Theta \leftrightarrow_{\Delta^t - S} \Theta'$ would be valid, a contradiction to the fact that (as assumed) Θ and Θ' are not equivalent (see Prop. 2.6).

Consider Ξ to be locally finite. The initial vertex v of Θ has an infinite transmitting area in Ξ (in fact, v is an initial vertex of Ξ). It follows from Lemma 2.30, that Ξ contains an ω-track, and this possibility has been excluded in the previous paragraph. It follows that Ξ contains a vertex x of infinite degree.

If Θ is an ω^*-track, then a dual argumentation is feasible, and one obtains a similar subgraph Ξ of Δ containing a vertex x of infinite degree.

From the construction of Ξ, it is easy to see that there exists a rest of Θ such that, for every vertex y of this rest, one has $x \leftrightarrow_{D^t} y$. Hence a rest of Θ belongs to a strong component of Δ^t which contradicts the acyclicity of $\mathcal{E}$. It follows the existence of a 1-track of Δ which belongs to $\mathcal{E}$ in Δ^t.

(3) '$\Leftarrow$' is a consequence of (1) '$\Leftarrow$' and (2) '$\Leftarrow$'. □

Corollary 2.32.
Let Δ be a digraph and $i \in \{\omega, \omega^*, c\}$.

(1) If all strong components of Δ are finite (e.g. if Δ is acyclic), then Δ is i-end-faithful with Δ^t.

(2) If Δ is locally finite, then Δ is i-endcomplete in Δ^t. □

Remark 2.33.
The above Theorem and its proof show that new i-ends in Δ^t occur if and only if Δ contains an infinite strong component without an i-end, whereas neither acyclic i-ends of Δ can melt together nor it is possible that there are acyclic i-ends in Δ^t that are not in Δ. Hence there is a natural bijection between the acyclic i-ends of Δ and the acyclic i-ends of Δ^t. On the other hand, different cyclic i-ends of Δ that belong to the same strong component of Δ, melt always together in Δ^t. Further, it is impossible that cyclic and acyclic i-ends of Δ melt together in Δ^t. □

The next theorem is a generalization of HALIN's Satz 3 in his paper [29] that every countable connected graph has an endfaithful spanning tree. Its proof relies heavily on HALIN's proof. Simple examples show that, as long as the proof remains unchanged, the assumption that the ω-ends of the digraph Δ are independent can not be omitted. On the other hand, up to now, the author was not able to find an example of a digraph which shows that this assumption is necessary. If it is possible at all then to get rid of this assumption one has either to adopt some *special* enumeration of the vertex set of Δ or to use a completely different method.

Theorem 2.34 (see [93], Satz 19).
If Δ is a countable digraph with an initial vertex such that all its ω-ends are independent then Δ contains an ω-endfaithful spanning arborescence. □

Proof:
Let Δ be a countable digraph with an initial vertex v such that all its ω-ends are independent. Since Δ is countable, it is possible to enumerate the vertices of Δ: $V(\Delta) = \{v_1, v_2, v_3, ...\}$, where, without loss of generality, $v_1 = v$.

Inductively, an increasing sequence $(\Lambda_i)_{i\in\mathbb{N}}$ of arborescences will be constructed such that v_1 is the initial vertex of every Λ_i, $i \in \mathbb{N}$, and the union of the arborescences yields an ω-endfaithful spanning arborescence of Δ. Since the weak components of a digraph can not necessarily be covered by the transmitting area of a single vertex and, further, the transmitting areas of different vertices can overlap, a sort of priority has to be given to the transmitting areas that occur in the subsequent induction. Let

$\Lambda_0 := \{\emptyset, \emptyset\}$,
$\Lambda_1 := \{v_1, \emptyset\}$,
$\Delta_1 := \Delta$,
$F_1 := \emptyset$, $L_1 := \emptyset$, $j_1 = 1$ and
$\Theta_1 := \{v_1, \emptyset\}$.

Assume that, for some $n \in \mathbb{N}$, the subarborescences Λ_1,..., Λ_n of Δ (all having v_1 as its initial vertex) are already constructed such that $\Lambda_1 \subset \Lambda_2 \subset ... \subset \Lambda_n$ and $v_i \in V(\Lambda_i)$ for all $i \in [n]$. Further, assume that, for every $2 \le i \le n$, j_i, Ψ_i, x_i, y_i, a_i, Θ_i, F_i, L_i, and Δ_i are determined according to the procedure in the next paragraph.

Now, let j_{n+1} the smallest number $\in \mathbb{N}$ such that $v_{j_{n+1}} \notin V(\Lambda_n)$. Clearly, $j_{n+1} \ge n+1$. Since v_1 is an initial vertex of Δ and of Λ_n, the set $\mathcal{S}$ of all $v_1 \to v_{j_{n+1}}$–tracks whose intersection with Λ_n is a track is nonempty. Now, choose a track $\Psi_{n+1} \in \mathcal{S}$ such that the terminal vertex x_{n+1} of $\Psi_{n+1} \cap \Lambda_n$ belongs to a Δ_i (for the definition of the Δ_i see below) with maximum $i \in [n]$ such that the $x_{n+1} \to v_{j_{n+1}}$–subtrack of Ψ_{n+1} is completely contained in Δ_i. Let y_{n+1} be the outer neighbour of x_{n+1} on Ψ_{n+1} and $a_{n+1} := (x_{n+1}, y_{n+1})$. Further, let

$\Theta_{n+1} := \Psi_{n+1} - \Lambda_n$,
F_{n+1} the transmitting area of y_{n+1} in $\Delta - A_n$,
$L_{n+1} := \{l \in [n] \mid F_{n+1} \not\subseteq F_l\}$, and
$\Delta_{n+1} := \Delta[F_{n+1}] - \bigcup_{m\in L_{n+1}} \Delta_m$.

This means that Δ_{n+1} is the subgraph of Δ that contains exactly the vertices that are reachable from y_{n+1} in $\Delta - \Lambda_n$ but do not belong to the transmitting area of some y_l, $l \in [n]$, in $\Delta - \Lambda_{l-1}$ which is not a superset of the transmitting area of y_{n+1} in $\Delta - \Lambda_n$. Now, let

$$\Lambda_{n+1} := \Lambda_n \cup \{a_{n+1}\} \cup \Theta_n \qquad (= \Lambda_n \cup \Psi_{n+1}).$$

Λ_{n+1} is an arborescence with the initial vertex v_1 and contains v_1,..., v_n, v_{n+1}. Now, consider

$$\Lambda := \bigcup_{i\in\mathbb{N}} \Lambda_i.$$

Obviously, Λ is a spanning arborescence of Δ and has v_1 as its initial vertex. Λ is the union of the pairwise disjoint $\Theta_n = \Lambda_n - \Lambda_{n-1}$ ($n \in \mathbb{N}$) and the arcs a_i, $i = 2, 3, ...$ From the fact that $F_n \subseteq F_l$ if and only if $y_n \in F_l$ it follows that

(1) $y_n \in V(\Delta_n)$ for all $n \in \mathbb{N}$.

(2) If $n < m \in \mathbb{N}$ and $V(\Delta_n) \cap V(\Delta_m) \neq \emptyset$, then $V(\Delta_m) \subset V(\Delta_n)$.

Assertion (2) will be shown by nested induction. Clearly, there is nothing to show if $n < m \leq 1$. Assume that the assertion is valid for every choice of $n < m \leq k \in \mathbb{N}$. It will be shown that then assertion (2) is also valid for every choice of $n < m \leq k+1$. Without loss of generality, let $m = k+1$. Clearly, assertion (2) is valid for $n = 1$, since $V(\Delta_m) \subset V(\Delta) = V(\Delta_1)$ (the subset relation is proper since $v_1 \notin V(\Delta_m)$). Now, for the second induction, let $l < k$ and assertion (2) be valid for all $n \leq l$. Then (2) is also true for $n = l+1$:

From $\Delta_n \cap \Delta_m \neq \emptyset$ it follows that $n \notin L_m$, i.e. $F_m \subseteq F_n$. If, for some $p \in L_n$, $\Delta_m \cap \Delta_p \neq \emptyset$ then $V(\Delta_m) \subset V(\Delta_p)$ because of the assumption of the second induction and $p \leq l$. But then it follows that $V(\Delta_m) \cap V(\Delta_n) \subseteq V(\Delta_p) \cap V(\Delta_n) = \emptyset$ since $p \in L_n$, a contradiction. Hence $\Delta_m \cap \left(\bigcup_{q \in L_n} \Delta_p\right) = \emptyset$. Together with $F_m \subseteq F_n$, this yields $V(\Delta_m) \subset V(\Delta_n) = \Delta[F_n] - \bigcup_{q \in L_n} \Delta_q$ (the subset relation is proper since A_k contains y_n; hence $y_n \notin V(\Delta_m)$, but $y_n \in V(\Delta_n)$ according to (1)). This completes both inductions.

(3) For all $n \in \mathbb{N}$, y_n is an initial vertex of Δ_n.

Otherwise, there is a vertex $a \in V(\Delta_n)$ which is not reachable from y_n in Δ_n. Since there exits a $y_n \to a$–track in $\Delta[F_n]$, this is possible only if there exists a vertex b on this track that belongs to some Δ_m, $m \in L_n$, and $a \notin V(\Delta_m)$. This leads into a recursion which ends in a contradiction since $m < n$ and n is finite.

(4) For all $n \in \mathbb{N}$, Θ_n is completely contained in Δ_n.

According to the definition, Θ_n is completely contained in the Δ_k with the biggest possible $k < n$. Because of (1), it follows that $\Delta_k \cap \Delta_n \neq \emptyset$. This and (2) yield $V(\Delta_n) \subset V(\Delta_k)$.

Consider $r \in L_n$ such that $V(\Theta_n) \cap V(\Delta_r) \neq \emptyset$. If $r > k$, then $v_{j_n} \notin V(\Delta_r)$ because of (3) and the maximality of k. It follows that $v_{j_n} \in V(\Delta_q)$ for some $q \in L_r$. Clearly, $q \leq k$ because of (3) and the maximality of k. Because of (2), it follows that $V(\Delta_r) \subset V(\Delta_k) \subseteq V(\Delta_q)$ which implies $F_r \subseteq F_q$, a contradiction to $q \in L_r$. Hence, $r \leq k$. (3) and $V(\Theta_n) \cap V(\Delta_r) \neq \emptyset$ yield $V(\Delta_n) \subset V(\Delta_k) \subseteq V(\Delta_r)$ which implies $r \notin L_n$, a contradiction. This proofs (4).

(5) For every $2 \leq n \in \mathbb{N}$, there exists a unique arc in Λ (namely a_n) which joins a vertex of Δ_n with a vertex of $\Delta - \Delta_n$.

If $V(\Delta_n) \cap V(\Theta_m) \neq \emptyset$ $(m \neq n)$ then $V(\Delta_m) \cap V(\Delta_n) \neq \emptyset$ because of (4). Clearly, $m > n$, and hence $\Theta_m \subset \Delta_m \subset \Delta_n$. Further, if $m \neq n$ and $\Theta_m \subset \Delta_n$ then $a_m \in A(\Delta_n)$. (This is due to the fact that x_m belongs to a Δ_k with the biggest possible $k \in [m-1]$ such that Θ_m is completely contained in Δ_k. It follows that $V(\Delta_k) \cap V(\Delta_n) \neq \emptyset$. It is impossible that $k < n$ since $m > n$ and k has been chosen maximally. Hence, $k \geq n$, which, together with (2), yields $\Delta_k \subseteq \Delta_n$. In particular, $a_m \in A(\Delta_n)$.) Clearly, it is impossible that there is an arc a_m having its endvertex $\in V(\Delta_n)$ such that Θ_m does not belong to Δ_n. Finally, there is no $x_m \in V(\Delta_n)$ such that $y_m \notin V(\Delta_n)$ because, if $x_m \in V(\Delta_n)$ then the Δ_k with the maximally chosen k such that it contains Θ_m completely and, in addition, x_m, is a subgraph of Δ_n as a consequence of $x_m \in \Delta_n$ and $k \geq n$.

(5) and the fact that every vertex of Δ_n is contained in some Θ_m (Λ is a spanning arborescence) yield

(6) $\Delta_n = \Delta[\bigcup_{k\in M_n} \Theta_k]$ where $M_n = \{s \in \mathbb{N} \mid v_{j_s} \in V(\Delta_n)\}$.

Now, it will be shown that Λ is ω-endcomplete in Δ. Let Π be any ω-track of Δ. Inductively, a decreasing sequence $\Delta_{n_1} \supset \Delta_{n_2} \supset \Delta_{n_3} \supset \ldots$ will be found such that $(n_i)_{i\in\mathbb{N}}$ is a strictly increasing sequence and, for all $i \in \mathbb{N}$, Δ_{n_i} contains a rest of Π. Set $n_1 = 1$. Clearly, $\Delta = \Delta_1$ contains Π completely. Let $\Delta_{n_1} \supset \ldots \supset \Delta_{n_m}$ be already constructed. Δ_{n_m} contains a rest Π' of Π and, therefore, infinitely many vertices. Thus, Θ_{n_m} does not exhaust Δ_{n_m}. (2) and (6) yield that $\{\Delta_r \mid \Delta_r \subset \Delta_{n_m}\} \neq \emptyset$, since $v(j_r) \in V(\Delta_{n_m})$ implies $r \geq n_m$. In particular, there exists a $\Delta_{l_1} \subset \Delta_{n_m}$ which contains a vertex w_1 of Π'. Δ_{l_1} contains a rest of Π except in the case that there exists $l_2 \in L_{l_1}$ such that $\Delta_{l_2} \subset \Delta_{n_m}$ contains a vertex w_2 of Π' which is behind w_1 on Π'. Δ_{l_2} contains a rest of Π except in the case that there exists $l_3 \in L_{l_2}$ such that $\Delta_{l_3} \subset \Delta_{n_m}$ contains a vertex w_3 of Π' which is behind w_2 on Π' etc. Since the l_i become smaller and smaller, the sequence ends at some $\Delta_{l_t} \subset \Delta_{n_m}$ that contains a rest of Π. Setting $n_{m+1} := l_t$ completes the induction.

For $i \in \mathbb{N} \setminus \{1\}$, let Υ_i denote the $v_1 \to y_i$–subtrack of Ψ_i (for the definition of the Ψ_i look at the construction of Λ at the beginning of this proof). Further, let $\Upsilon := \bigcup_{i\in\mathbb{N}} \Upsilon_{n_i}$. If $i < j$ then $\Upsilon_{n_i} \subset \Upsilon_{n_j}$ since $y_{n_j} \in V(\Delta_{n_j}) \subset V(\Delta_{n_i})$ and there exists a unique $v_1 \to z$–track in Λ for each $z \in V(\Lambda)$, and because of (5). As the union of a strictly increasing sequence of infinitely many tracks with the same initial vertex v_1 Υ is an ω-track with the initial vertex v_1. Υ contains all y_{n_i}, $i \in \mathbb{N}$. Hence, Π stays reachable by Υ after the removal of any finite vertex set from Δ because of (3), i.e. $\mathcal{E}_\Delta(\omega,\Upsilon) \geq \mathcal{E}_\Delta(\omega,\Pi)$. Thus, $\mathcal{E}_\Delta(\omega,\Upsilon) = \mathcal{E}_\Delta(\omega,\Pi)$, since the ω-ends of Δ are pairwise independent. Because Π has been chosen arbitrarily, Λ contains at least one ω-track from each ω-end of Δ, i.e. Λ is endcomplete in Δ.

Λ is also ω-endinjective in Δ. Let Π and Π' be any pair of ω-tracks of Λ (without loss of generality, both with v_1 as their initial vertex) such that Π and Π' belong to the same end of Δ. It has to be shown that Π and Π' are also end-equivalent in Λ. First of all, let $\Delta_{n_1} \supset \Delta_{n_2} \supset \Delta_{n_3} \supset \ldots$ be a sequence as above such that each Δ_{n_i}, $i \in \mathbb{N}$, contains a rest of Π. By induction, it will be shown that

(7) for all $n \in \mathbb{N}$, Δ_n either contains a rest of both Π and Π' or neither a rest of Π nor a rest of Π'.

The statement is trivial for $n = 1$. Assume (7) to be valid for every $i \in [n]$. Consider Δ_{n+1} and let $m \leq n$ be largest possible such that Δ_m contains a rest of Π and Π', respectively. Without loss of generality, assume that Δ_{n+1} contains a rest of Π. This and (2) imply that $V(\Delta_{n+1}) \subset V(\Delta_m)$. If $l \in L_{n+1}$ and Δ_l contains a rest of Π *or* of Π', then Δ_l contains a rest of *both* Π and Π' because of the assumption of the induction. But then Δ_{n+1} contains neither a rest of Π nor a rest of Π', a contradiction. Note, that if one of the Δ_t, $t \in L_{n+1}$, contains a vertex of Π, then there is some $s \in L_{n+1}$ such that Δ_s contains a rest of Π. Further, since Π and Π' belong to the same end of Δ, Proposition 2.6 yields the existence of infinitely many pairwise disjoint $\Pi \to \Pi'$–tracks as well as infinitely many pairwise disjoint $\Pi' \to \Pi$–tracks. The removal of Λ_n destroys only finitely many of these tracks. All this together yields that

Δ_{n+1} contains also a rest of Π', and the induction step is complete.

(7) yields that every Δ_{n_i} contains also a rest of Π'. (5) yields that both Π and Π' contain all a_{n_i}, $i \in \mathbb{N} \setminus \{1\}$ and thus belong to the same ω-end of Λ which completes the proof. □

The next statement gives a sufficient condition for a digraph to have no comparable ω-ends, thus yielding the subsequent corollary.

Proposition 2.35.
If Δ is a strongly connected, inner locally finite digraph and does not contain an ω^**-track, then all* ω*-ends of* Δ *are independent.* □

Proof:
Consider $\mathcal{E} < \mathcal{F}$ to be two comparable ω-ends of Δ and $\Theta \in \mathcal{E}$, $\Psi \in \mathcal{F}$. Let v be the initial vertex of Θ and — beginning with the initial vertex — $w_1, w_2, w_3,$... the vertices of Ψ. Since Δ is strongly connected, there exists a $w_i \to v$–track Π_i for every $i \in \mathbb{N}$. As Δ is inner locally finite, according to Lemma 2.30, $\Xi := \bigcup_{i\in\mathbb{N}} \Pi_i$ contains an ω^*-track, a contradiction. □

Corollary 2.36.
Every countable, inner locally finite, strongly connected digraph without ω^**-tracks has an end-faithful spanning arborescence.* □

2.5 Geodesic, Infinite Tracks in Digraphs

In the sequel, some results of M. E. WATKINS [87] are generalized to digraphs. The main result of this section, Theorem 2.40, will be needed in some subsequent investigations. Let Θ be an arbitrary (finite or infinite) track of Δ. Θ is called *geodesic* if, for all $u \leq_\Theta v$, $\Theta[u,v]$ is a shortest $u \to v$-track in Δ.

The next result is obtained by the application of a technique of KÖNIG to digraphs ([51], p. 81, Satz 6; see also Lemma 2.30).

Proposition 2.37.
In an outer (inner) locally finite digraph exactly the vertices with an infinite transmitting (receiving) area lie on a geodesic ω*- (*ω^**-) track, respectively.* □

Proposition 2.38.
In an infinite, connected, v-transitive digraph Δ *all vertices have an infinite transmitting as well as an infinite receiving area.* □

Proof:
Suppose $v_0 \in V(\Delta)$ to have a finite transmitting area S in Δ, $|S| = n$ say. Then, for all $v \in S$, $V_\Delta(v \to) = S$, since $V_\Delta(v \to) \subseteq S$ and $|V_\Delta(v \to)| = |S| = n$. ($\Delta$ is v-transitive!) But since Δ is connected and infinite, there exist vertices $w \notin S$, $w' \in S$, such that (w, w') is an arc of Δ. It follows $|V_\Delta(w \to)| \geq n + 1$, contradicting the v-transitivity of Δ. □

Simple examples show that Proposition 2.38 is not valid for almost v-transitive digraphs. An immediate consequence of the last two results is the following

Corollary 2.39.
In an infinite, connected, v-transitive digraph Δ every vertex lies both on an ω- and an ω^*-track. In particular, Δ has at least one c-end, one ω-end, and one ω^*-end. If, in addition, Δ is outer (inner) locally finite, then every vertex lies on a geodesic ω- (ω^*-) track, respectively. □

The above results solely serve the purpose of the preparation of the following

Theorem 2.40.
In an infinite, connected, v-transitive digraph Δ every vertex lies on a 2-track. If, in addition, Δ is locally finite, every vertex lies on a geodesic 2-track. □

Proof:
As Δ is v-transitive, Δ is (r,s)-regular for some cardinals r,s. If both r and s are finite, that is, Δ is locally finite, then it is possible to apply a slightly changed version of a proof of THOMASSEN and WOESS ([80], Proposition 5.2). If both r and s are infinite, then it is easy to construct a 2-track inductively. If r is uncountable and s is finite (or vice versa), then it is possible to start with an ω- (ω^*-) track and to extend it to a 2-track inductively.

It remains the case where $r = \aleph_0$ and s is finite (or vice versa). Without loss of generality, let Δ be an infinite, connected, v-transitive, $(\aleph_0,s)$-regular digraph ($s \in \mathbb{N}$). Choose an arbitrary vertex $v_0 \in V(\Delta)$. According to Corollary 2.39, there exists a geodesic ω-track Θ_0 possessing v_0 as its initial vertex. In turn, let v_1, v_2, v_3,... denote the other vertices of Θ_0.

Assume that all the inneighbours of v_0 are on Θ_0. For all $n \in \mathbb{N}$, there exists $\sigma_n \in Aut(\Delta)$ such that $\sigma_n(v_n) = v_0$. Let $\Theta_n := \sigma_n(\Theta_0)$ and $M := \{\sigma_n(v_0) \mid n \in \mathbb{N}, (v_n, v_0) \in A(\Delta)\}$. Clearly, for every $u \in M$, there exists $n \in \mathbb{N}$ such that $\sigma_n((v_n, v_0)) = (v_0, u) \in A(\Delta)$. Since v_0 has finite outdegree, M is finite. Hence there exist $w \in M$, $k,l \in \mathbb{N}$ such that the $w \to v_0$-tracks on Θ_k and Θ_l are of different lengths. This contradicts the fact that Θ_k and Θ_l are geodesic ω-tracks.

Therefore, there is a vertex v_{-1} lying not on Θ_0 such that $(v_{-1}, v_0) \in A(\Delta)$. Let $\Theta_0^{(1)} := \Theta_0 \cup \{v_{-1}\} \cup \{(v_{-1}, v_0)\}$. Now, assume that all the inneighbours of v_{-1} are on $\Theta_0^{(1)}$. For each $n \in \mathbb{N}$, there exists $\sigma_n^{(1)} \in Aut(\Delta)$ such that $\sigma_n^{(1)}(v_n) = v_{-1}$. Let $\Theta_n^{(1)} := \sigma_n^{(1)}(\Theta_0^{(1)})$ and $M^{(1)} := \{\sigma_n^{(1)}(v_{-1}) \mid n \in \mathbb{N}, (v_n, v_{-1}) \in A_\Delta\}$. Clearly, for every $u \in M^{(1)}$, there exists $n \in \mathbb{N}$ such that $\sigma_n^{(1)}((v_n, v_{-1})) = (v_{-1}, u) \in A_\Delta$. Since v_{-1} has finite outdegree, $M^{(1)}$ is finite. Hence, there exist $w^{(1)} \in M^{(1)}$ such that $w^{(1)}$ is the initial vertex of infinitely many of the $\Theta_n^{(1)}$. Since $w^{(1)}$ has finite outdegree, there exists an outneighbour $x^{(1)}$ of $w^{(1)}$ such that $x^{(1)}$ is the second vertex on infinitely many of the $\Theta_n^{(1)}$. Hence there exist infinitely many $x^{(1)} \to v_{-1}$-tracks of different lengths, a contradiction to the fact that every $\Theta_n^{(1)}$ is from its second vertex on a geodesic. Now it is easy to see that a 2-track can be constructed inductively. □

The example $DK_{\aleph_0}$ shows that in a connected, v-transitive digraph of infinite degree a *geodesic* 2-track can not be expected.

Theorem 2.41.
Let Δ be an infinite, connected, locally finite, v-transitive digraph and Θ be a geodesic 2-track of Δ. Then $\Delta - \Theta$ has no finite components. □

Proof:
Δ^u is infinite, connected, locally finite and v-transitive. According to Theorem 4.2 of [87], $\Delta^u - \Theta^u$ has no finite component. Hence $\Delta - \Theta$ has no finite component. □

2.6 Ends in Vertex-Transitive Digraphs and Automorphisms

2.6.1 Ends and Automorphisms of Undirected Graphs

In the sequel, some notions and results concerning ends and automorphism of undirected graphs are listed and commented.

Definition 2.42 (translation, τ-periodic).
An automorphism τ of a (di)graph Δ is called a *translation* if $\tau[F] \neq F$ for all $F \sqsubseteq V(\Delta)$. If τ is a translation of Δ and Θ is an infinite 2-ray (2-track), then Θ is called τ-*periodic* if there exists $k \in \mathbb{N}$ such that $\tau^k(\Theta) = \Theta$.

The following theorem justifies in a way the naming "translation" in giving a very detailed description of a graph Γ which admits a translation. If τ is a translation of Γ, then there is a fixed end under τ which is uniquely determined by the sequences $(\tau^i(v))_{i\in\mathbb{N}_0}$ $(v \in V(\Gamma))$ in the sense of the proof of Theorem 4 in [34]. This end will be called the *direction of* τ and will be denoted by $\mathcal{D}_\tau$. For the definition of m_1, see Section 2.2.

Theorem 2.43. *(*HALIN **73, [34],** Theorem 9, Corollary 14**;** DIESTEL/JUNG/ MÖLLER **93, [16]***)*

Let τ be a translation of the connected graph Γ. Then the following statements are valid.

(1) $m_1(\mathcal{D}_\tau) = m_1(\mathcal{D}_{\tau^{-1}})$.

(2) $\mathcal{D}_\tau \neq \mathcal{D}_{\tau^{-1}}$ *if and only if* $m_1(\mathcal{D}_\tau) < \infty$.

(3) There exists a system of $m_1(\mathcal{D}_\tau)$ *pairwise disjoint τ-periodic 2-rays in Γ.*

(4) If $\mathcal{D}_\tau = \mathcal{D}_{\tau^{-1}}$, then every finite set of pairwise disjoint τ-periodic 2-rays can be extended to an infinite set of pairwise disjoint τ-periodic 2-rays.

(5) If $m_1(\mathcal{D}_\tau) = m < \infty$, then $n \in \mathbb{N}$, $S \sqsubseteq V(\Gamma)$, a 'section graph' Γ_0, and pairwise disjoint τ-periodic 2-rays Π_1, ..., Π_m can be found such that

(a) $\tau^n[\Pi_i] = \Pi_i$ *for all* $i \in [m]$,

(b) S separates $\mathcal{D}_\tau$ and $\mathcal{D}_{\tau^{-1}}$,

(c) S consists of m vertices, hence meets each Π_i in exactly one vertex t_i,

(d) the subgraphs $\tau^{nj}[S]$, $j \in \mathbb{Z}$, are pairwise disjoint,

(e) if $v \in V(\Pi_i)$, $w \in V(\Pi_j)$, and $v < t_i$, $t_j < w$, where $<$ denotes the natural order of Π_i (or Π_j, respectively) given by τ, then S separates v and w in Γ; hence

(f) if $i < j < k$ are arbitrary integers, then $\tau^{jn}[S]$ separates in Γ each vertex of $\tau^{in}[S]$ from each vertex of $\tau^{kn}[S]$, and no pair of vertices of a $\tau^{jn}[S]$ is separated in Γ by any other $\tau^{in}[S]$.

(g) $\Gamma = \bigcup_{j\in\mathbb{Z}} \tau^{jn}[\Gamma_0]$, *and, for all $k \in \mathbb{Z}$,*

$$\left(\bigcup_{i=-\infty}^{j-1} \tau^{in}[\Gamma_0]\right) \cap \tau^{jn}[\Gamma_0] = \tau^{jn}[S].$$

Further, if Γ_0 contains a ray Ψ, then the $\tau^j[\Psi]$, $j \in \mathbb{Z}$, are pairwise not equivalent to each other in Γ and are not elements of $\mathcal{D}_\tau \cup \mathcal{D}_{\tau^{-1}}$. If Γ_0 is finite, then $\mathcal{D}_\tau \neq \mathcal{D}_{\tau^{-1}}$ and there are no other ends in Γ. □

Corollary 2.44 (HALIN **73, [34],** Corollary 15)**.**
If τ is a translation of the connected graph Γ, then Γ has either 1, 2, or infinitely many ends. If $|\Omega_\Gamma| < \infty$, then $|\Omega_\Gamma| = 1$ if and only if $m_1(\Gamma) = \infty$, and $|\Omega_\Gamma| = 2$ if and only if $m_1(\Gamma) < \infty$. □

The following seminal result of JUNG was a milestone in the investigation of infinite, v-transitive graphs.

Theorem 2.45 (JUNG **81, [44],** Theorem 1)**.**
If Γ is a graph, G is a v-transitive subgroup of $Aut(\Gamma)$, and Y is an infinite fragment of Γ, then there exists $\sigma \in G$ such that $\sigma[Y \cup \partial Y] \subseteq Y$. □

Later, JUNG characterized proper translations in the following way. (A *proper translation* τ is a translation which fixes exactly two ends which, according to Theorem 2.43, means that $\mathcal{D}_\tau \neq \mathcal{D}_{\tau^{-1}}$.)

Theorem 2.46 (JUNG **91, [45],** Theorem 2.7)**.**
An automorphism τ of a connected graph Γ is a proper translation if and only if there exists an induced connected subgraph Ξ of Γ and $n \in \mathbb{N}$ such that $0 < |\partial\Xi| < \infty$ and $\tau^n[\Xi \cup \partial\Xi] \subseteq \Xi$. □

Hence, Theorem 2.45 says that every v-transitive, infinite, connected graph with an infinite fragment admits a proper translation and, therefore, has at least two ends. If τ is a proper translation, then every τ-periodic 2-track is a $\mathcal{D}_\tau, \mathcal{D}_{\tau^{-1}}$-track.

Further, JUNG and WATKINS were able to show that a v-transitive, connected, infinite graph with more than two ends must have at least infinitely many ends, as can be seen from the next theorem which has to prepared by some new terminology. A set $\mathcal{M}$ of fragments is called *strongly disjoint* if, for all $X, Y \in \mathcal{M}$, $(X \cup \partial X) \cap (Y \cup \partial Y) = \emptyset$.

Definition 2.47 (strip, directed strip).
A *strip* is a locally finite, connected graph with exactly two ends which admits a translation. A *directed strip* is a locally finite, connected digraph admitting a translation such that the underlying undirected graph has exactly two ends. □

Theorem 2.48 (JUNG/WATKINS **84, [50],** Corollary 4.4)**.**
If Γ is a connected, v-transitive graph with $\kappa_\infty(\Gamma) < \infty$ and not a strip, then each infinite fragment of Γ contains an infinity of strongly disjoint infinite fragments. □

The next theorem is an important result concerning infinite v-transitive graphs and has been a sort of folklore in infinite graph theory (at least in its locally finite version) for several years. (The locally finite version was implicitly contained in Theorem 2.45 and Theorem 2.48 and can even traced back to Theorem 10 of [34].) Using independent methods, MOHAR [54] proved the same result for connected, locally finite, almost v-transitive, infinite graphs.

Theorem 2.49 (DIESTEL/JUNG/MÖLLER **93, [16],** Corollary 4)**.**
A v-transitive, connected, infinite graph has either 1, 2, or at least $2^{\aleph_0}$ ends. □

DIESTEL, JUNG, and MÖLLER derived this result with the aid of the following interesting theorem and the subsequent corollary. If Γ is an infinite, connected graph and G is a subgroup of $Aut(\Gamma)$, then it is said that G *shuffles* Γ if, for every $S \sqsubseteq V(\Gamma)$ and every infinite $M \subseteq V(\Gamma)$ with finite boundary, there exists a $\sigma \in G$ such that $\sigma[S] \subseteq M$.

Theorem 2.50 (DIESTEL/JUNG/MÖLLER **93, [16],** Theorem 3)**.**
If Γ is a connected graph of infinite diameter, then Γ is shuffled by any group $G \leq Aut(\Gamma)$ that acts transitively on $V(\Gamma)$. □

Corollary 2.51 (DIESTEL/JUNG/MÖLLER **93, [16],** Corollary 2)**.**
If Γ is a v-transitive, connected graph of infinite diameter, $M \subseteq V(\Gamma)$ is infinite, and ∂M is finite, then M has infinite diameter in Γ. □

Further, DIESTEL, JUNG, and MÖLLER showed

Theorem 2.52 (DIESTEL/JUNG/MÖLLER **93, [16],** Theorem 7)**.**
If Γ is a v-transitive, connected graph with exactly two ends, then Γ is locally finite. (In fact, Γ is a strip.) □

The assertion in the brackets can immediately be seen from the proof of Theorem 7 in [16].

Every v-transitive, infinite, connected graph with more than one end admits a translation. The seemingly not yet solved question occurs whether this is also true in the 1-ended case.

Problem 7.
Does every 1-ended, connected, v-transitive graph admit a translation? □

To find a counterexample it may be worth to search among the connected, infinite, locally finite graphs whose automorphism group contains a v-transitive torsion subgroup. Examples of such graphs are provided by CAYLEY graphs of finitely generated, infinite torsion groups whose existence has been shown by GOLOD and SHAFAREVITCH [23]. Some results about the connectivity of such graphs are due to BABAI and WATKINS [8].

2.6.2 Ends and Automorphisms of Digraphs

In this section the attempt is made to carry over as much as possible from the results of the previous section to digraphs.

JUNG ([45], Proposition 2.6) gave a very short proof of a version of the following statement for undirected graphs. The proof given here bases on JUNG'S proof.

Proposition 2.53.
Let Δ be a connected digraph and $\tau \in Aut(\Delta)$ be a translation of Δ. Then Δ contains a nonempty set of τ-periodic 2-tracks if and only if there exists $k \in \mathbb{Z} \setminus \{0\}$ and $v \in V(\Delta)$ such that there is a $v \to \tau^k(v)$-track in Δ.

Proof:
The 'only if'-part is trivial. To show the 'if'-part, let

$$\Psi \in \bigcup_{v \in V(\Delta)} \{\Theta \mid \exists k \in \mathbb{Z} \setminus \{0\} : \Theta \text{ is a } v \to \tau^k(v)\text{-track}\} \neq \emptyset$$

be shortest possible. Then Ψ is a $w \to \tau^n(w)$-track, say. For $j \in [|n|]$, let

$$\Psi_j := \bigcup_{i \in \mathbb{Z}} \tau^{in+j}(\Psi).$$

It will be shown that every Ψ_j, $j \in [|n|]$, is a τ-periodic 2-track in Δ. For $i \in \mathbb{Z}$ and $j \in [|n|]$, let $\Psi_{ij} := \tau^{in+j}(\Psi - \tau^n(w))$. $\Xi := \Psi_{ij} \cap \Psi_{i'j'} \neq \emptyset$ for $i \neq i'$ or $j \neq j'$ is impossible: Let $x \in V(\Xi)$ and $k = i'n + j' - (in + j)$. Then there exists either a $\tau^k(x) \to x$–subtrack or an $x \to \tau^k(x)$–subtrack of $\Psi_{i'j'}$ which would be shorter than Ψ, contradicting the minimality of Ψ. It follows that $(\Psi_{ij} \cup \{\tau^{(i+1)n+j}(w)\}) \cap \Psi_{i'j'}$ contains exactly the initial of $\Psi_{i'j'}$ — which is also the terminal of Ψ_{ij} — if and only if $j = j'$ and $i' = i+1$. $\square$

The next result formulates a part of HALIN'S above mentioned result in an appropriate manner for digraphs. Note that $Aut(\Delta) \subseteq Aut(\Delta^u)$ for every digraph Δ.

Theorem 2.54.
Let Δ be a digraph and τ be a translation of Δ such that τ is a proper translation of Δ^u, i.e. τ fixes exactly two ends of Δ^u, namely $\mathcal{E} := \mathcal{D}_\tau$ and $\mathcal{E}' := \mathcal{D}_{\tau^{-1}}$. Let r be the maximum cardinality of a system of pairwise disjoint τ-periodic $\mathcal{E}' \to \mathcal{E}$–tracks in Δ, s be the maximum number of pairwise disjoint ω-tracks in Δ such that the underlying undirected rays are in $\mathcal{E}$, and s^* be the maximum number of pairwise disjoint ω^*-tracks in Δ such that the corresponding underlying rays are in $\mathcal{E}'$. Then $r = s = s^*$. $\square$

Proof:
Let n, m, S and $(\Delta^u)_0$ be as in Theorem 2.43 (with respect to τ). Clearly, $r \leq s, s^*$. Let Θ_1,..., Θ_s be a system of pairwise disjoint ω–tracks such that Θ_1^u,..., Θ_s^u are in $\mathcal{E}$. There exists $k \in \mathbb{N}$ such that, for all $l \geq k$, $\tau^{nl}[S] \cap \Theta_i \neq \emptyset$ for every $i \in [s]$. Since S and s are finite, there exists a sequence $k < p_1 < p_2 < p_3 < ...$ of integers such that, for all $l \in [s]$ and whenever $i < j$, all vertices of $\tau^{np_i}[S] \cap \Theta_l$ are on Θ_l *before* the vertices of $\tau^{np_j}[S] \cap \Theta_l$.

Now, an *intersection pattern* of $N_i := \tau^{np_i}[S]$ with $\bigcup_{l \in [s]} \Theta_l$ will be understood as a 4-dimensional boolean $(m,m,s,9)$-matrix M_i. Consider the vertices of S to be arbitrarily labeled with the integers 1,...,m, and the vertices of each N_i to be labeled according to the induced labeling of $\tau^{ni}[S]$. Further, note that if v is a vertex of $N_i \cap \Theta_l$, then there are exactly nine possible ways for Θ_l to pass through v (see Fig. 2.5).

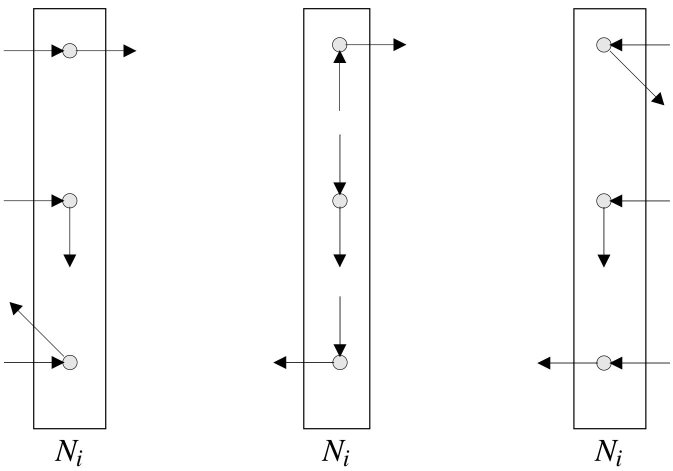


Figure 2.5: The nine ways some Θ_l can pass through a vertex v of some N_i.

Consider these nine ways to be denoted by w_1,..., w_9, respectively. Now, an entry m^i_{abcd} of M_i is defined to be 1 if and only if the vertex v_a of N_i is — in the order of the vertices of $N_i \cap \Theta_c$ on Θ_c — the bth vertex on Θ_c, and Θ_c passes v_a in the way w_d. Otherwise, m^i_{abcd} is defined to be 0.

Since S is finite, there are only finitely many intersection patterns. It follows that there are $i < j$ such that N_i and N_j have the same intersection pattern with $\bigcup_{l \in [s]} \Theta_l$. It follows, that the translation of

$$\Upsilon := \bigcup_{i \in [s]} \Theta_i \cap \left(\bigcup_{i=p_i}^{p_j - 1} \tau^{ni}[(\Delta^u)_0] \right)$$

by $\tau^{n(p_j - p_i)}$ generates at least s τ-periodic 2-tracks. A dual proof holds for the statement about ω^*-tracks. Therefore, $s = r = s^*$. □

An automorphism φ of the (di)graph Δ is called *unbounded* if $\{dist(v, \varphi(v)) \mid v \in V(\Delta)\}$ is unbounded in $\mathbb{N}$.

Corollary 2.55.
If a (directed) strip Σ admits an unbounded automorphism, then $r = r^* = s = s^* = t = t^*$, where r^*, t, and t^* are the corresponding parameters to r, s, and s^* in $\mathcal{E}'$, $\mathcal{E}'$ and $\mathcal{E}$, respectively. □

Proof:
Let φ be an unbounded automorphism of Σ. Because of Theorem 5.10 (b) in [50], which says that an automorphism ψ of a strip Σ is not unbounded if and only if, for any fragment C of Σ, $C \cap \psi[C]$ and $\overline{C} \cap \psi[\overline{C}]$ are fragments while $C \cap \psi[\overline{C}]$ and $\overline{C} \cap \psi[C]$ are not fragments, and due to the fact that every fragment of Σ^u corresponds to one of the ends $\mathcal{E}$, $\mathcal{E}'$ of Σ^u, φ exchanges the ends $\mathcal{E}$ and $\mathcal{E}'$ of Σ^u. Clearly, the ω-tracks whose underlying rays belong to $\mathcal{E}$ are mapped by φ onto ω-tracks whose underlying rays belong to $\mathcal{E}'$. □

The next statement is due to the fact that there is no finite, v-transitive, acyclic digraph except the digraph which consists of a single vertex. Remember, that Δ^* is always acyclic.

Proposition 2.56.
A v-transitive, connected digraph either is strongly connected or has infinitely many strong components. All these strong components are v-transitive and isomorphic. □

The following theorem is a slightly incomplete version of Theorem 2.49 for digraphs.

Theorem 2.57.
If Δ is an infinite, connected, v-transitive digraph, then the following statements hold:

(1) If Δ^u has one end, then Δ has

- *infinitely — more precisely, at least the number of strong components of Δ times the number of c-ends of such a strong component — many composed ends provided that Δ is not strongly connected and any strong component of Δ has at least one c-end;*
- *either two or infinitely many c-ends provided that Δ is not strongly connected and no strong component of Δ has a c-end.*

(2) If Δ^u has two ends, then Δ is a directed strip. Every v-transitive, directed strip has exactly two c-ends.

(3) If Δ is locally finite or strongly connected, and Δ^u has at least $2^{\aleph_0}$ ends, then Δ has at least $2^{\aleph_0}$ c-ends. □

Proof:
ad (1): Let Δ^u have only one end. From Proposition 2.56 it follows that Δ either is strongly connected or has infinitely many v-transitive strong components that are all isomorphic. The former case will not be examined here.

In the latter case, if one of the strong components has a c-end, then Δ has infinitely many c-ends by Lemma 2.9. If none of the strong components has a c-end, then, by Theorem 2.14, Δ can be assumed to be acyclic. According to Theorem 2.40, every vertex lies on a 2-track. Since there are no acyclic proper c-ends (Proposition 2.11), Δ has at least two c-ends. Consider that, for every vertex of Δ, its transmitting area has only one ω-end and its receiving area has only one ω^*-end. As Δ is connected, there is a zigzag-track between any pair u, v of vertices in Δ which means that one can find a finite sequence of vertices with u as its first vertex and v as its last vertex such that the transmitting areas of consecutive vertices in the sequence do overlap. The intersection of two overlapping transmitting areas contains the transmitting area of some vertex w which contains at least one ω-end. It follows that Δ has exactly one ω-end because every ω-track is contained in the transmitting area of some vertex of Δ. Similarly, one shows that Δ has exactly one ω^*-end. Altogether, Δ has exactly two c-ends. Therefore, if Δ has more than two c-ends, then the transmitting area of any vertex of Δ has two ω-ends or the receiving area of any vertex of Δ has two ω^*-ends. Without loss of generality, assume that every transmitting area has at least two ω-ends. Let $\mathcal{E}$ and $\mathcal{E}_0$ be two ω-ends of the transmitting area of the vertex v_0 and $\Theta \in \mathcal{E}$, $\Theta_0 \in \mathcal{E}_0$ be two ω-tracks. Without loss of generality, assume $\Theta \not\leq \Theta_0$. Then Θ contains a vertex v_1 whose transmitting area does not intersect Θ_0. Since Δ is v-transitive, the transmitting area of v_1 contains an ω-track Θ_1 that belongs to an ω-end $\mathcal{E}_1 \neq \mathcal{E}$. Clearly, also $\mathcal{E}_1 \neq \mathcal{E}_0$. Now, it is easy to see that one can construct infinitely many ω-ends of Δ by induction.

ad (2): If Δ is a digraph, then $Aut(\Delta)$ is a subgroup of $Aut(\Delta^u)$. Hence it follows by Theorem 2.45 and Theorem 2.46 that every v-transitive, connected digraph Δ such that Δ^u is 2-ended is a (directed) *strip*.

According to Theorem 2.40, Δ contains a 2-track Θ. Let τ be a translation of Δ and $\mathcal{E}$, $\mathcal{F}$ be the ends of Δ^u. As $V(\Delta)$ has only finitely many $< \tau >$-orbits, at least one of these orbits has infinitely many vertices in common with Θ. By Proposition 2.53, it follows the existence of at least one τ-periodic 2-track Ψ in Δ. Remember that Ψ^u is an $\mathcal{E},\mathcal{F}$–path. Hence every vertex v of Δ lies on a 2-track Θ_v such that $(\Theta_v)^u$ is an $\mathcal{E},\mathcal{F}$–path, since Δ is v-transitive. Let $(\Delta^u)_0$, S, and n be as in Theorem 2.43 (5) (with respect to τ). For any vertex v of $(\Delta^u)_0$, there are integers $k < 0 < l$ such that $\tau^k[S]$ has the same intersection pattern (see the proof of Theorem 2.54) with Θ_v as $\tau^l[S]$ in such a way that all vertices of $\tau^k[S] \cap \Theta_v$ are before (or behind) all vertices of $\tau^l[S] \cap \Theta_v$. Now, it is easy to see that v lies on a τ-periodic 2-track. Since $(\Delta^u)_0$ is finite, and because $(\Delta^u)_0$ intersects with all $< \tau^n >$-orbits, only finitely many τ-periodic 2-tracks are necessary to cover all vertices of Δ.

Let Θ_1,..., Θ_m be such a system of τ-periodic 2-tracks. Choose Θ_i and Θ_j arbitrarily. Assume Δ to contain a $\Theta_i \to \Theta_j$–track Π but no $\Theta_j \to \Theta_i$–track. Then $\Theta_i < \Theta_j$ since the Θ_n are τ-periodic. Let u be a vertex of Θ_i, v be a vertex of Θ_j, and σ be an automorphism of Δ such that $\sigma(u) = v$. Such a σ exists since Δ is v-transitive. Let Ψ be an ω-subtrack of $\sigma[\Theta_i]$. Ψ has no vertex in common with Θ_i, because, otherwise, Θ_i is reachable by Θ_j. Hence Ψ has infinitely many vertices in common with one of the Θ_n, $n \neq i$, Θ_k say. Let Ψ' denote a sub-1-track of $\sigma[\Theta_j]$ such that $\Psi < \Psi'$. Ψ' has in common infinitely many vertices with one of the Θ_n, Θ_l say. It follows $\Theta_j \leq \Theta_k$ and $\Theta_k < \Theta_l$, thus $\Theta_j < \sigma[\Theta_j]$. (Otherwise, $\Psi < \Psi'$ is impossible.) Now it is easy to see that an infinite chain of disjoint 2-tracks can be constructed inductively, contradicting the fact that Δ contains only finitely many pairwise disjoint 2-tracks.

Now consider that neither Θ_i is reachable by Θ_j nor vice versa. As Δ is connected, there exists a zigzag-track Υ joining Θ_i and Θ_j. Let z be the vertex of Υ adjacent to the rim vertex of Υ which is on Θ_j. Without loss of generality, assume that the arc a between z and Θ_j is directed towards Θ_j. z lies on one of the Θ_n, Θ_q say. It follows $\Theta_q \leq \Theta_j$, since there are infinitely many disjoint images of a from Θ_q to Θ_j because Θ_q and Θ_j are both τ-periodic. It follows $\Theta_q \sim \Theta_j$ since $\Theta_q < \Theta_j$ is impossible. Now it is easy to see that $\Theta_i \sim \Theta_j$ follows by induction. Every 1-track in Δ has infinitely many vertices in common with some sub-1-track of some Θ_n.

ad (3): At first, assume Δ to be strongly connected. As Δ^u has more than two ends, there exists $S \sqsubset V(\Delta)$ such that $\Delta^u - S$ has at least three infinite components. Denote two of these components with Ξ_0 and Ξ_1, respectively. Clearly, S can be chosen such that $\Delta[S]$ is strongly connected. Because of Theorem 2.50, one can find automorphisms σ_i such that $S_i := \sigma_i[S] \subseteq \Xi_i$ $(i = 0, 1)$. As Δ is strongly connected, there exists an $S \to S_i$–track in Δ $(i = 0, 1)$. Now, it is easy to see that one can inductively construct an ω-endinjective subdivision of the dyadic arborescence in Δ. It follows that Δ has at least $2^{\aleph_0}$ c-ends.

Now, if Δ is not strongly connected, then it has infinitely many v-transitive strong components Ξ_i, $i \in I$, which are all isomorphic. If Ξ is such a strong component and Ξ^u has at least $2^{\aleph_0}$ many ends, then it is possible to use the argumentation in the previous paragraph. If Ξ^u has one or two ends, and there are at least $2^{\aleph_0}$ strong components, then, of course, Δ has at least $2^{\aleph_0}$ c-ends.

It will be shown that, if Ξ is finite or if Ξ^u has one or two ends and Δ has $< 2^{\aleph_0}$ strong components, then $(\Delta^*)^u$ has at least $2^{\aleph_0}$ ends.

Consider a ray Π in Δ^u to have infinitely many vertices in common with Ξ^u, and denote these vertices with v_1, v_2, v_3,... according to their order on Π. Then Ξ^u is not finite and, according to the assumption, has one or two ends. If Ξ^u has one end and is locally finite, then the graph obtained by the union of arbitrarily chosen $v_i \to v_{i+1}$-tracks in Ξ^u $(i \in \mathbb{N})$ is locally finite and, therefore, contains a ray. This ray is end-equivalent with Π. If Ξ^u has one end and infinite degree, then pick up some ray Π' in Ξ^u. No vertex x of Ξ^u can be separated from Π' by a finite subgraph Υ of Ξ^u since, otherwise, the component of $\Xi^u - \Upsilon$ which contains x contains a ray — which can easily be constructed by induction since Ξ^u has infinite degree —, contradicting the fact that Ξ^u has only one end. Hence it is possible to construct infinitely many pairwise disjoint Π, Π'-paths in Ξ^u by induction. If Ξ^u has two ends, then Ξ^u is a strip (Theorem 2.52). Now, it is very easy to see that Π is end-equivalent with some ray in Ξ^u.

In the last paragraph, it has been shown that a ray which has an infinite intersection with some $(\Xi_j)^u$, $j \in I$, is contained in an end which also contains a ray of $(\Xi_j)^u$. Let f be the function that maps a subgraph Υ of Δ^u onto the subgraph of $(\Delta^*)^u$ which has as its vertices the $(\Xi_i)^u$ that have a nonempty intersection with Υ and which contains an edge $\{(\Xi_i)^u, (\Xi_j)^u\}$ whenever $i \neq j$ and there is an edge in Υ that connects $(\Xi_i)^u$ with $(\Xi_j)^u$. Clearly, every ray of Δ^u whose intersection with every $(\Xi_i)^u$ is finite is mapped by f onto a ray of $(\Delta^*)^u$. If two such rays Π and Π' belong to different ends of Δ^u, then $f[\Pi]$ and $f[\Pi']$ belong to different ends of $(\Delta^*)^u$ since any finite set separating Π and Π' in Δ^u is mapped by f onto a finite set of $(\Delta^*)^u$ which separates $f[\Pi]$ and $f[\Pi']$. It follows that $(\Delta^*)^u$ has at least $|\Omega_{\Delta^u}| - |I||\Omega_{(\Xi_i)^u}| = 2^{\aleph_0}$ ends (for any $i \in I$), since either both $|I|$ and $|\Omega_{(\Xi_i)^u}|$ are less than $2^{\aleph_0}$ or $|\Omega_{(\Xi_i)^u}| = 0$.

This yields that if Δ is not strongly connected, then Δ can be assumed to be acyclic because of Corollary 2.15.

Now, let Δ be locally finite and assume that, for some $v \in V(\Delta)$, neither a pair of ω-tracks of the transmitting area of v nor a pair of ω^*-tracks of the receiving area of v can be separated by some $S \sqsubseteq V(\Delta)$. A similar argumentation as in (1) using the v-transitivity and connectedness of Δ yields that no pair of ω-tracks and no pair of ω^*-tracks in Δ can be separated in Δ by a finite $S \sqsubseteq V(\Delta)$. It follows that, for every $S \sqsubseteq V(\Delta)$, $\Delta - S$, and hence $\Delta^u - S$, has at most two infinite components, a contradiction to the fact that Δ^u has more than two ends. Hence some pair of ω-tracks in the transmitting area of v or some pair of ω^*-tracks in the receiving area of v can be separated by some $S \sqsubseteq V(\Delta)$. Without loss of generality, assume the former to be true. Let Θ_0 and Θ_1 be a pair of ω-tracks with the same initial vertex v such that, for some $S \sqsubseteq V(\Delta)$, $rest(\Theta_0 - S)$ and $rest(\Theta_1 - S)$ are in different components Υ_0 and Υ_1 of $\Delta - S$, respectively.

According to Corollary 2.51, both $(\Upsilon_0)^u$ and $(\Upsilon_1)^u$ have infinite diameter in Δ^u. As Δ is locally finite, the distances of the vertices of Θ_0 and Θ_1 from S become arbitrarily large. Hence there exist automorphisms ψ_i, $i \in \mathbb{N}$, such that $\psi_i(v) \in V(\Theta_0)$ and $\psi_{i+1}[S]$ is completely contained in the component of $\Delta - \psi_i[S]$ which contains $rest(\Theta_0 - \psi_i[S])$ for every $i \in \mathbb{N}$.

Now, let the vertices of Θ_0 — beginning with the initial vertex — be denoted by v, $u_{0,1}$, $u_{0,2}$, $u_{0,3}$,..., and, for $i \in \mathbb{N}$, let $\varphi_{0,i}$ be an automorphism of Δ with $\varphi_{0,i}(v) = u_{0,i}$. Consider that

(*) there is no ω-track Θ_{01} in Υ_0 which has its initial vertex on Θ_0 such that rests of Θ_0 and Θ_{01} can be separated in Δ by some $F \sqsubset V(\Delta)$.

It follows that, for all $i \in \mathbb{N}$, S intersects with $\Psi_{0,i}$, where $\Psi_{0,i} = \varphi_{0,i}[\Theta_0]$ or $\Psi_{0,i} = \varphi_{0,i}[\Theta_1]$, and that, for all $i \in \mathbb{N}$, rests of $\Psi'_{0,i}$ and Θ_0 are in the same component of $\Delta - \varphi_{0,1}[S]$, where $\Psi'_{0,i} = \varphi_{0,i}[\Theta_k]$ if $\Psi_{0,i} = \varphi_{0,i}(\Theta_{1-k})$. For all $i \in \mathbb{N}$, all but finitely many of the $\Psi_{0,j}$, $j \in \mathbb{N}$, intersect with $\psi_i[S]$. Hence, with the aid of the $\Psi_{0,j}$, $j \in \mathbb{N}$, one easily derives the existence of an ω^*-track $\Pi_1 \subseteq \Upsilon_0 \cup S$ such that $\Theta_0 \leq \Pi_1$. (This would also be valid if Δ had infinite degree since the $\psi_i[S]$, $i \in \mathbb{N}$ have all the same finite cardinality.) It follows that $\Theta_0 < \Pi_1$ because Δ is acyclic. Let the vertices of Π_1 — beginning with the terminal vertex — be denoted by $u_{1,1}$, $u_{1,2}$, $u_{1,3}$,..., and, for every $i \in \mathbb{N}$, let $\varphi_{1,i}$ be an automorphism such that $\varphi_{1,i}(v) = u_{1,i}$. Because of (*), it follows that, for all $i \in \mathbb{N}$, S intersects with $\Psi_{1,i}$, where $\Psi_{1,i} = \varphi_{1,i}[\Theta_0]$ or $\Psi_{1,i} = \varphi_{1,i}[\Theta_1]$, and that, for all $i \in \mathbb{N}$, rests of $\Psi'_{1,i}$ and Θ_0 are in the same component of $\Delta - \varphi_{1,i}[S]$, where $\Psi'_{1,i} = \varphi_{1,i}[\Theta_k]$ if $\Psi_{1,i} = \varphi_{1,i}(\Theta_{1-k})$. With the aid of the $\Psi'_{1,i}$, $i \in \mathbb{N}$, one easily derives the existence of an ω-track $\Pi_2 \subseteq \Upsilon_0$ such that $\Pi_1 \leq \Pi_2$. It follows that $\Pi_1 < \Pi_2$ since Δ is acyclic.

Repeating this procedure, one obtains an infinite sequence $\Pi_1 < \Pi_2 < \Pi_3 < ...$ of 1-tracks such that each Π_i, $i \in \mathbb{N}$, intersects with all but finitely many of the $\psi_j[S]$, $j \in \mathbb{N}$. This contradicts the fact that S is finite.

Hence assumption (*) must be false. It is easy to see that if (*) is false, then one can construct a subdivision of the dyadic arborescence which is ω-endinjective in Δ. □

To complete Theorem 2.57, one has to deal with the conjecture below. Beforehand, some arguments are given to justify this conjecture.

If Δ is a strongly connected, v-transitive digraph such that Δ^u has exactly one end then it seems that, due to the v-transitivity of Δ, the strongly connectedness of Δ is distributed uniformly over the whole of Δ, i.e., for every $S \sqsubseteq V(\Delta)$, there exists $S' \sqsubseteq V(\Delta)$ such that $S \subseteq S'$ and $\Delta - S'$ is strongly connected. If so, then it is easy to see that Δ has one end.

If Δ is a v-transitive, connected digraph such that Δ^u has at least $2^{\aleph_0}$ ends, then even in the case that Δ is neither locally finite nor strongly connected it seems that Δ contains a 1-track Θ such that the distances between any vertex of Δ and the vertices of Θ become arbitrarily large.

Conjecture 2.58.
Let Δ be an infinite, v-transitive, connected digraph. Then the following statements are valid:

(1) If Δ^u has exactly one end and Δ is strongly connected, then Δ has exactly one c-end.

(2) If Δ^u has at least $2^{\aleph_0}$ ends, then Δ has at least $2^{\aleph_0}$ c-ends, even if Δ is not locally finite or strongly connected. □

The example of the alternating oriented 2-ray shows that an arc-transitive digraph need not to contain even a 1-track.

2.7 Digraphs Without 1-Tracks

In this section a theorem very similar to Satz 1 of Halin [30] is proved.

Let Δ be a digraph. An infinite $M \subseteq V(\Delta)$ is called *bilaterally dispersed* if there exists an $S \sqsubseteq V(\Delta)$ such that every unilateral component of $\Delta - S$ contains only finitely many vertices of M. Note, that unilateral components are not necessarily disjoint.

Theorem 2.59.
Let Δ be a digraph. Then the following statements are equivalent:

(1) Δ contains no 1-track.

(2) For all infinite $M \subseteq V(\Delta)$, there exists an infinite, bilaterally dispersed $M' \subseteq M$. □

Proof:

$\Leftarrow$: If Δ contains a 1-track Θ, then set $M := V(\Theta)$ and let $T \sqsubseteq V(\Delta)$ and M' an arbitrary infinite subset of M. Then $rest(\Theta - T)$ is unilaterally connected, i.e. it belongs to a unilateral component of $\Delta - T$, and, of course, contains infinitely many vertices of M', a contradiction. Hence, Δ contains no 1-tracks.

$\Rightarrow$: Assume Δ to contain no 1-tracks and that

(I) there exists an infinite $M \subseteq V(\Delta)$ such that, for all $T \sqsubseteq V(\Delta)$ and for all infinite $M' \subseteq M$, there exists a unilateral component Ξ of $\Delta - T$ such that Ξ contains infinitely many vertices of M'.

Every unilateral component of a digraph represents a maximal chain of strong components with respect to the reachability relation in Δ^*. The existence of a unilateral component Ξ containing infinitely many strong components implies the existence of a strictly increasing or decreasing countable sequence of strong components (with regard to reachability). With the aid of these strong components one can very easily construct an ω- or ω^*-track (using the same technique as in Proposition 2.12), a contradiction to the assumption. Hence, every unilateral component of Δ contains only finitely many strong components. Therefore, (I) can be transformed to

(II) There exists an infinite $M \subseteq V(\Delta)$ such that, for all $T \sqsubseteq V(\Delta)$ and for all infinite $M' \subseteq M$, there exists a *strong* component Ξ of $\Delta - T$ such that Ξ contains infinitely many vertices of M'.

Now, it is possible to apply the technique HALIN used in his proof of Satz 1 in [30]: Let Ξ_0 be a strong component containing infinitely many vertices of $M_0 := M$ and v_0 be an arbitrarily chosen vertex from Ξ_0. Because of (II), there exists a strong component Ξ_1 of $\Xi_0 - v_0$ containing an infinite $M_1 \subseteq \Xi_0 \cap M_0$. (For the strong components of $\Xi_0 - v_0$ are also strong components of $\Delta - v_0$. This is not necessarily true for unilateral components!) There exists a $v_0 \to \Xi_1$-track Π_1. Let v_1 denote the terminal of Π_1.

If $M_0, \ldots, M_n$, $\Xi_0, \ldots, \Xi_n$, $v_0, \ldots, v_n$ and $\Pi_1, \ldots, \Pi_n$ are already constructed such that, for all $i \in [n]$, Ξ_i is a strong component of $\Xi_{i-1} - v_{i-1}$, Π_i is a $v_{i-1} \to \Xi_i$-track, v_i is the terminal of Π_i, $M_i \subseteq M_{i-1}$, and Ξ_i contains infinitely many elements from M_i, then, because of (II), there exists a strong component Ξ_{n+1} of $\Xi_n - v_n$ containing an infinite $M_{n+1} \subseteq \Xi_n \cap M_n$. (The strong components of $\Xi_{n+1} - v_n$ are also strong components of $\Delta - \bigcup_{i \in [n] \cup \{0\}} \{v_i\}$, since $v_0, \ldots, v_{n-1}$ are not contained in Ξ_n.) In Ξ_n there exists a $v_n \to \Xi_{n+1}$-track Π_{n+1}. Let v_{n+1} denote the terminal of Π_{n+1}.

By induction, there exist infinitely many tracks Π_n ($n \in \mathbb{N}$) such that $\Pi_i \cap \Pi_j = \emptyset$ for $|i - j| > 2$ and $\Pi_i \cap \Pi_{i+1} = v_i$ for $i \in \mathbb{N}$. Hence, the union of the Π_i ($i \in \mathbb{N}$) is an ω-track, a contradiction. Therefore, (I) must be false, and the theorem is proved. □

Chapter 3

The Classification of All Vertex-Transitive, Planar Strips

3.1 Preliminaries, Part 2

Definition 2.47 introduces the concept of a strip. To a wide extent, the following terminology is taken over from THOMASSEN [77]. A (finite or infinite) graph $\mathfrak{G}$ is called a *plane graph* if it is drawn in the Euclidean plane such that two edges of $\mathfrak{G}$ never have a point in common — except possibly an endvertex (see HARARY [38]). An abstract graph Γ is called *planar* if it is isomorphic to some plane graph $\mathfrak{G}$, and any such plane graph $\mathfrak{G}$ is called an *embedding of* Γ. A cycle $\mathfrak{C}$ of a plane graph $\mathfrak{G}$ is called a *facial cycle*, and the interior (resp. exterior) of $\mathfrak{C}$ is called a *face of* $\mathfrak{G}$ iff the interior (resp. exterior) of $\mathfrak{C}$ does not intersect $\mathfrak{G}$. The set of all faces of a plane graph $\mathfrak{G}$ is denoted by $F(\mathfrak{G})$. A *k-face* is a face which is incident with k vertices. If f is a face of a plane graph $\mathfrak{G}$, then the vertex set of the facial cycle related to f may also be denoted by $V(f)$. Two faces of a plane graph $\mathfrak{G}$ are called *adjacent* if their related facial cycles have an edge in common. A plane graph $\mathfrak{G}$ is called *f-transitive* if $Aut(\mathfrak{G})$ acts transitively on the set of all facial cycles of $\mathfrak{G}$. A plane graph $\mathfrak{G}'$ is called a *(geometric) dual (of the plane graph $\mathfrak{G}$)* iff there is a bijection $\varphi : V(\mathfrak{G}') \to F(\mathfrak{G})$ such that, for all $v, w \in V(\mathfrak{G}')$, v lies in the face $\varphi(v)$ and, for every common edge e of the faces $\varphi(v)$ and $\varphi(w)$ in $\mathfrak{G}$, $\mathfrak{G}'$ contains an edge vw_e that intersects e but no other edge of $\mathfrak{G}$, and there are no other edges in $\mathfrak{G}'$. Thus a dual can contain loops and multiple edges. A plane graph is called *self-dual* if it is isomorphic to its dual.

If it is said that some sequence occurs *in the cyclic ordering* of A around B, then this means that those sequence occurs in the clockwise *or* in the counterclockwise cyclic ordering of A around B. The one- (two-) dimensional topological sphere will be denoted by S^1 (S^2), respectively.

Remark 3.1.
If v_1v_2, v_2v_3,..., v_nv_1 are, in the clockwise ordering, the edges of a facial cycle, then $v_{i-1}v_i$ and v_iv_{i+1} are consecutive and $v_{i-1}v_i$ comes before v_iv_{i+1} for all $i \in [n]$ (in the counterclockwise ordering of the edges around v_i; indices mod n). □

The following characterization of (finite and infinite) planar graphs from WAGNER [85] used DIRAC's and SCHUSTER's extension [17] of the famous Theorem of KURATOWSKI [52] to countable graphs. An account on KURATOWSKI's Theorem is due to THOMASSEN [78].

Theorem 3.2 (WAGNER **67, [85],** 1.1)**.**
A graph Γ is planar if and only if

(1) Γ has at most $2^{\aleph_0}$ vertices,

(2) Γ has at most countably many vertices of degree at least 3, and

(3) Γ does not contain a subdivision of a K_5 or a $K_{3,3}$. □

Condition (3) of Theorem 3.2 could also be formulated as follows:

(3') Both K_5 and $K_{3,3,}$ are not minors of Γ.

This is an immediate consequence of

Proposition 3.3 (DIESTEL **96, [15],** Proposition 3.4.2)**.**
A graph Γ contains K_5 or $K_{3,3}$ as a minor if and only if Γ contains a subdivision of a K_5 or a $K_{3,3}$. □

The following result was first proved for the finite case by WHITNEY [89].

Theorem 3.4 (IMRICH **75, [41]).**
Every 3-connected, planar, finite or infinite graph has a unique embedding into the Euclidean plane in the sense that — up to a possible simultaneous reversal of the edge-orderings at all vertices — the cyclic order of the edges around a vertex is the same in every plane embedding. □

Note, that the graphs in Theorem 3.4 are allowed to have infinite degree. Furthermore, the facial cycles of these graphs are the same for all embeddings into S^2. It follows that one can speak of "the" facial cycles, "the" faces, and "the" dual of such a graph.

Proposition 3.6 is a version of the next statement, which WATKINS [88] needed to classify all e-transitive, planar strips.

Proposition 3.5 (WATKINS **91, [88],** Lemma 2.5)**.**
If Γ is a planar graph, F is a finite set of vertices of Γ, Ξ_1, Ξ_2 are distinct components of $\Gamma - F$, $x \in F$, and v_i, $u_i \in V(\Xi_i)$ $(i = 1,2)$ are adjacent to x in Γ, then, for no planar embedding of Γ, the neighbours of x (in the counterclockwise cyclic ordering) can form the cyclic subsequence u_1, u_2, v_1, v_2. □

Proposition 3.6.
If $\mathfrak{G}$ is a plane graph, $\mathfrak{C}$ is a facial cycle of $\mathfrak{G}$ with u_1, v_1, u_2, $v_2 \in V(\mathfrak{C})$, $\mathfrak{P}_1$ is a u_1,v_1-path and $\mathfrak{P}_2$ is a u_2,v_2-path of $\mathfrak{G}$, and $\mathfrak{P}_1$ and $\mathfrak{P}_2$ are disjoint, then the cyclic ordering of the vertices of $\mathfrak{C}$ cannot contain the subsequence u_1, u_2, v_1, v_2. □

Proof:
Follows immediately from Proposition 3.5 by adding a new vertex x to $\mathfrak{G}$ such that x lies in the face whose border is $\mathfrak{C}$ and is adjacent to u_1, u_2, v_1, v_2, and by deleting all vertices ($\neq x$) from $\mathfrak{G}$ that do not belong to $V(\mathfrak{P}_1) \cup V(\mathfrak{P}_2)$. □

Let Γ be a 3-connected, planar graph. An edge of Γ is called a *k,l-edge* if it is incident with a k-face and an l-face of Γ. Γ is said *to have the vertex type* $(a_1.a_2.\ldots.a_n)$ $(n \in \mathbb{N})$ if Γ is

n-regular and, for every vertex $v \in V(\Gamma)$, one can find a sequence of faces in the cyclic ordering of the faces around v such that the first member of the sequence is an a_1-face, the second one an a_2-face,..., and the last one an a_n-face. A vertex type of the form $(\dots.b.b.\dots.b.\dots)$, where there are m consecutive b's, may be written $(\dots.b^m.\dots)$. By convention, one starts with the smallest possible value for a_1 and takes the smallest possible value for the next a_j as long as possible. (For example, (5.5.5.3.4.4) is also written $(3.4^2.5^3)$, and (4.6.4.4.3) is also written $(3.4^2.6.4)$.)

Let Γ be a v-transitive graph and G be a subgroup of $Aut(\Gamma)$. An *imprimitivity system* I *of* G *on* Γ is a partition of $V(\Gamma)$ into so-called *blocks* such that G acts on I.

Proposition 3.7.
If Γ is a graph and I is an imprimitivity system of Γ whose blocks induce connected subgraphs of Γ, then the following statements are valid:

(1) If Γ is v-transitive, then Γ_I is v-transitive.

(2) If Γ is planar, then Γ_I is planar. □

Proof:
The first statement is trivial. The second statement follows from Theorem 3.2 and Proposition 3.3. □

3.2 The Classification

The aim of this section is to give a complete classification of all v-transitive, planar strips. For that purpose, one needs some information about the structure of v-transitive strips. The following result by IMRICH and SEIFTER [42], which is an immediate consequence of the Theorem 2.43 of HALIN, provides some useful properties of v-transitive strips:

Proposition 3.8 (IMRICH/SEIFTER **88/89, [42],** Proposition 3.2)**.**
If Σ is a v-transitive strip, then $V(\Sigma)$ is spanned by a system $\mathcal{S} = \{\Pi_1, ..., \Pi_m\}$ of $m = \kappa_\infty(\Sigma)$ pairwise disjoint 2-rays, and there exists a translation τ of Σ such that $\tau[\Pi_i] = \Pi_i$ for all $i \in [m]$. □

In the sequel, any spanning system of $\kappa_\infty(\Sigma)$ many pairwise disjoint 2-rays in a v-transitive strip Σ such that each of the 2-rays is left invariant by the same translation τ of Σ is called a *main system of* Σ *(with respect to* τ*)*. Note that if $\mathcal{E}$ and $\mathcal{F}$ are the ends of the strip Σ, then every 2-ray of Σ that is left invariant by some translation of Σ is an $\mathcal{E},\mathcal{F}$-ray.

It immediately follows that there is only one v-transitive strip with $\kappa_\infty = 1$, namely the 2-ray. BABAI [7] gives a classification of all *stripes*, which he defines as v-transitive graphs with 2 ends and connectivity exactly 2. There are exactly 6 of them (see Fig. 3.1). It turns out that all stripes are planar and admit a translation, such that a stripe is also a strip. Note that a v-transitive strip with $\kappa_\infty = 2$ has connectivity exactly 2, which can be checked easily in the light of Proposition 3.8. Hence Fig. 3.1 shows all v-transitive, planar strips with $\kappa_\infty = 2$. This means that, in the sequel, it can be assumed that $\kappa_\infty \geq 3$.

Let $\mathcal{E}$ and $\mathcal{F}$ be two distinct ends of a graph Γ. Furthermore, let Π and Π' be two disjoint $\mathcal{E},\mathcal{F}$-rays of Γ, and let uu', vv' be a pair of disjoint edges of Γ such that $u, v \in V(\Pi)$

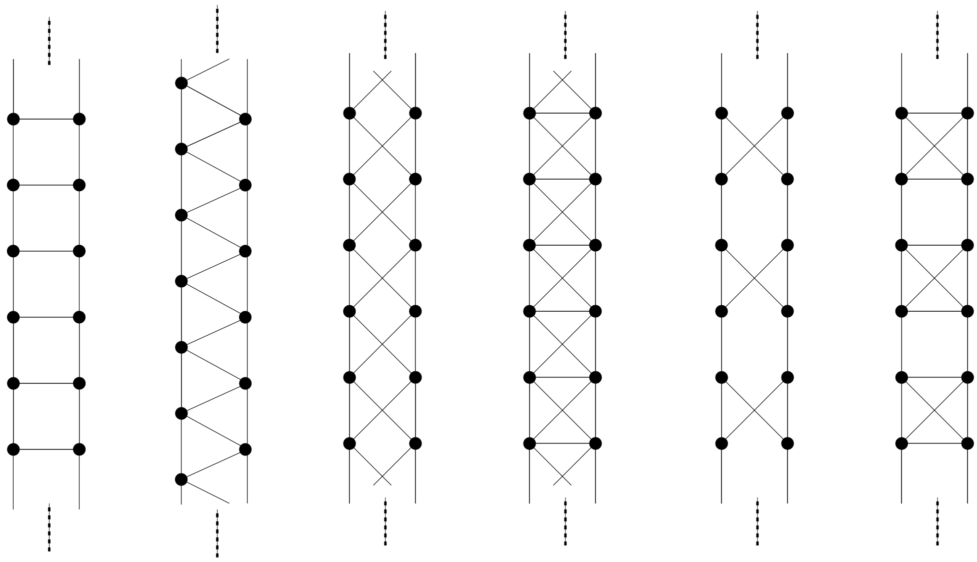


Figure 3.1: The six v-transitive, planar strips of connectivity 2. (a) The two-way infinite ladder. (b) The two-way infinite antiprism. (c) The crossed strip. (d) The crossed ladder. (e) The alternating crossed strip. (f) The alternating crossed ladder.

and $u', v' \in V(\Pi')$. uu' and vv' are called *a pair of crossing edges (between* Π *and* Π'*)* if $v \in V(\Pi[u,\mathcal{E}])$ and $v' \in V(\Pi[u',\mathcal{F}])$. uu' and vv' are called *a pair of consecutive edges (between* Π *and* Π'*)* if $v \in V(\Pi[u,\mathcal{F}])$ and $v' \in V(\Pi[u',\mathcal{F}])$ and, for every edge xx' between Π and Π', either $x \in V(\Pi[u,\mathcal{E}])$ and $x' \in V(\Pi'[u',\mathcal{E}])$ or $x \in V(\Pi[v,\mathcal{F}])$ and $x' \in V(\Pi'[v',\mathcal{F}])$.

The next result implies the embeddability of a v-transitive, planar strip with $\kappa_\infty \geq 3$ on a 2-way infinite cylinder. Any such embedding can be transformed to an embedding in the Euclidean plane by 'going into the inner space of the cylinder and looking down' (see, e.g., Fig. 3.5, p. 65).

Lemma 3.9.
If Σ *is a v-transitive, planar strip with* $m := \kappa_\infty(\Sigma) \geq 3$ *and* $\mathcal{S} = \{\Pi_1, ..., \Pi_m\}$ *is a main system of* Σ*, then*

(1) $\Sigma_{\mathcal{S}}$ *is a cycle of length m,*

(2) there is no pair of crossing edges between any pair Π_i*,* $\Pi_j \in \mathcal{S}$*,*

(3) Σ *is 3-connected, and*

(4) Θ *is a facial cycle of* Σ *if and only if there exists* $i \in [m]$ *and a pair* uu'*,* vv' *of consecutive edges between* Π_i *and* Π_{i+1} *such that* $\Theta = \Pi_i[u,v] \cup uu' \cup \Pi_{i+1}[u',v'] \cup vv'$ *(indices mod m). Furthermore, no facial cycle separates* Σ*.* □

Proof:
Let $\mathcal{S}$ be a main system of Σ with respect to the translation τ of Σ.

(1) The proof of the first statement is divided into three steps.

(a) Σ_S has minimal degree 2.

Clearly, Σ_S cannot contain isolated vertices, since strips are connected. Hence, assume that there is a vertex of degree 1 in Σ_S. Without loss of generality, consider the vertices of Π_1 to be adjacent only to vertices of Π_1 and Π_2. If all edges between Π_1 and Π_2 would belong to other edge orbits than the edges of Π_1 and Π_2, then there were an automorphism φ such that $\varphi[\Pi_2] = \Pi_1$ (Σ is v-transitive!) and $\varphi[\Pi_1] \cap V(\Pi_2) \neq \emptyset$. Obviously, there would exist vertices of $\varphi[\Pi_1]$ being adjacent to vertices of some $\varphi[\Pi_j]$, $j \neq 1,2$, a contradiction. Hence, at least one edge $e = uv$ between Π_1 and Π_2 belongs to a $\psi[\Pi_i]$, for some $\psi \in Aut(\Sigma)$ and $i \in [m]$. There exists a minimal end-separating set S of m vertices in Σ. Since Σ is v-transitive, every vertex of Σ is in such a set. Clearly, it is impossible for u and v to be together in such a set S'. Otherwise, there would exist $k \in [m]$ such that $S' \cap \psi[\Pi_k] = \emptyset$, a contradiction.

There exists an edge wx such that $w \in V(\Pi_1(u,\mathcal{E}))$ and $x \in V(\Pi_2(v,\mathcal{F}))$ (or vice versa), where $\mathcal{E}$ and $\mathcal{F}$ are the two ends of Σ. This is due to the fact that $v \in V(\Pi_2)$ belongs to an end-separating set S''. S'' contains exactly one vertex of each Π_i. If an edge like e' would not exist, then the vertex which S'' has in common with Π_1 could be replaced by u, since there are no edges between any Π_l, $l > 2$, and Π_1, and, in addition, the Π_i are induced subgraphs of Σ (otherwise an end-separating set had to contain more than m vertices). It has already been shown that u and v cannot be together in a set like S''.

The reader may note that, up to this point, planarity has not been used. This will be done now. Since all Π_i are τ-invariant, Σ contains a subdivision of the graph shown in Fig. 3.2. The same figure shows that this graph, and hence Σ, contains a subdivision of a $K_{3,3}$, which contradicts the planarity of Σ. Thence, (a) must be valid. Note, that this part also includes the Proof of (2).

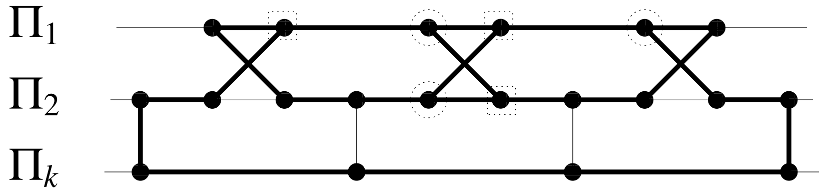


Figure 3.2: The thick lines yield a subdivision of a $K_{3,3}$. The bipartition of the main vertices of the subdivision is shown by dotted boxes resp. cycles.

(b) Σ_S does not contain a vertex of degree ≥ 3.

Consider Σ_S to contain a vertex Π_i of degree ≥ 3. It will be shown that, under this assumption, Σ contains a subdivision of a $K_{3,3}$. Let Π_j, Π_k, and Π_l be three neighbours of Π_i in Σ_S. Since Σ is a strip, it is easy to find a subdivision Ξ' of the graph Ξ consisting of two 4-cycles glued together along one edge in Σ such that three main vertices of Ξ' belong to Π_i and the other three main vertices of Ξ' belong to Π_j, and both Π_i and Π_j contain one main vertex of degree 3 of Ξ'. Note, that the vertex set of Ξ can be partitioned into two sets of three pairwise independent vertices. To obtain a subdivision of a $K_{3,3}$, one needs only two more paths. These can be found easily by using Π_k for one of these paths and Π_l for the other of these paths.

(c) Now, assertion (1) follows immediately from (a), (b), and the fact that Σ is connected.

(2) has been shown at the end of the proof of (1)(a).

(3) Choose any pair of vertices x, y. The degree of Σ is at least three. Since $\Sigma_{\mathcal{S}}$ is a cycle of length at least three and there are infinitely many edges between 2-rays that are adjacent in $\Sigma_{\mathcal{S}}$, it is easy to find three inner vertex-disjoint x,y-paths in Σ, even if x and y are on distinct 2-rays $\in \mathcal{S}$.

(4) '$\Leftarrow$': Let uu', vv' be a pair of consecutive edges between Π_i and Π_{i+1} and let Θ be defined as in the theorem. Clearly, Θ is a cycle. Σ can be embedded into the surface of the two-way infinite cylinder $S^1 \times \mathbb{R} \subset \mathbb{R}^3$ in the following manner. Embed the Π_i, $i \in [m]$, in such a way that the canonical projection of $S^1 \times \mathbb{R}$ onto $S^1 \times \{0\}$ reduces the embedded Π_i, $i \in [m]$, to single points. Because of (2), embedding the remaining edges of Σ onto geodetic arcs of $S^1 \times \mathbb{R}$ yields an embedding $\mathfrak{S}$ of Σ such that the interior of Θ does not intersect with the embedding of Σ. This embedding can be transformed into a plane embedding by "going into the inner space of the cylinder and looking down". Hence, Θ is a facial cycle of Σ.
'$\Rightarrow$': Now, if Θ is a facial cycle of Σ, then it has some vertex x in common with some Π_i, $i \in [m]$. If one walks on Π_i along the edges of Θ in the clockwise order, towards the end $\mathcal{E}$ of Σ say, one reaches a first edge uu' leaving Π_i, since Π_i is induced (possibly $u = x$). Without loss of generality, let this edge be between Π_i and Π_{i+1} (indices mod m). Because of Remark 3.1, the next edge $u'w$ of Θ either goes back to Π_i or is an edge of Π_{i+1}. If it goes back to Π_i, then $w \in V(\Pi_i(u,\mathcal{F}))$ because of Remark 3.1. As Θ is a cycle, one even has $w \in V(\Pi_i[x,\mathcal{F}))$ ($w \neq x$, if $u = x$). If $u'w$ is an edge of Π_{i+1}, then, because of Remark 3.1, $w \in V(\Pi_{i+1}(u',\mathcal{F}))$. Walking further along the edges of Θ in the clockwise order, one reaches a first edge $v'v$ leaving Π_{i+1}. Because of Remark 3.1, this edge is an edge between Π_{i+1} and Π_i. Clearly, $v \in V(\Pi_i[x,\mathcal{F}))$ since Θ is a cycle, since there is no pair of crossing edges between Π_i and Π_{i+1}, and because of Remark 3.1. If $v \neq x$, then one reaches x by walking further along the edges of Θ in the clockwise order. None of these edges can leave Π_i since Θ is a cycle, since there is no pair of crossing edges between Π_i and Π_{i+1}, and because of Remark 3.1. Clearly, uu' and vv' (resp. $u'x$) are a pair of consecutive edges between Π_i and Π_{i+1}.

It follows that no facial cycle does separate Σ since $m \geq 3$. □

From Lemma 3.9 (3) it follows, that every v-transitive, planar strip Σ with $\kappa_\infty \geq 3$ has a unique plane embedding in the sense of Lemma 3.4. Furthermore, it follows that Σ has (up to isomorphism) a unique dual Σ^*, and that the cyclic order of the faces around a vertex is the same at every vertex (which means that Σ has a vertex type), since Σ is v-transitive. For the classification, it will be essential to determine all the vertex types a planar, v-transitive strip can have. For this purpose, the reader should remember EULER's Formula:

$$|V| - |E| + |F| = 2, \tag{3.1}$$

for every finite, connected, planar multigraph Γ, where $|F|$ denotes the number of faces of any plane representation of Γ (see, e.g., HALIN [37]).

Now, consider that the finite, connected, plane multigraph $\mathfrak{G}$ of degree ρ has a vertex type, $(n_1.n_2.\dots.n_\rho)$ say. It follows

$$|E| = \frac{\rho}{2}|V| \quad \text{and} \tag{3.2}$$

$$|F| = \sum_{k=1}^{\rho} \frac{|V|}{n_k} = |V| \sum_{k=1}^{\rho} \frac{1}{n_k}. \tag{3.3}$$

Putting (3.2) and (3.3) into (3.1), one gets

$$|V| - \frac{\rho}{2}|V| + |V| \sum_{k=1}^{\rho} \frac{1}{n_k} = 2. \tag{3.4}$$

From this, one easily deduces

$$\sum_{k=1}^{\rho} \frac{n_k - 2}{n_k} = 2 - \frac{4}{|V|}. \tag{3.5}$$

To make this formula (or a similar one) fertile for *infinite* graphs, one has to be very careful, as has been pointed out by GRÜNBAUM and SHEPHARD [24]. Consider a 3-connected, planar strip Σ which has a vertex type and admits the translation τ. For every subgraph Ξ of Σ, call $v \in V(\Xi)$ a *rim vertex of* Ξ if there is a facial cycle Υ of Σ which includes v such that Υ is not a subgraph of Ξ. Choose Σ_0 and n according to Theorem 2.43 and let $\Sigma_1 \subset \Sigma_2 \subset \Sigma_3 \subset \dots$ be the increasing sequence of subgraphs of Σ where $\Sigma_i = \Sigma_{i-1} \cup \tau^{in}[\Sigma_0] \cup \tau^{-in}[\Sigma_0]$ for all $i \in \mathbb{N}$. The number of rim vertices of the Σ_i, $i \in \mathbb{N}$, is bounded, since the facial cycles of Σ have a bounded length and $\tau^{in}[\Sigma_0] \cap \tau^{(i+1)n}[\Sigma_0]$ is a κ_∞-seperator of Σ for every $i \in \mathbb{Z}$. It follows that the Σ_i have an increasing 'core' of vertices with the same vertex type. Hence one has to take into account only some *bounded* loss of edges/faces in equation (3.2)/ (3.3), respectively, such that in equation (3.5) only the numerator of the last fraction differs and is bounded (if worked out for the Σ_i). It follows that, for determining the possible vertex types which a planar strip Σ can have, one has the formula

$$\sum_{k=1}^{\rho} \frac{n_k - 2}{n_k} = 2. \tag{3.6}$$

Surprisingly (or not surprisingly), this is the same formula as equation (2.1.2) in [25], which was deduced by GRÜNBAUM and SHEPHARD to obtain all possible arrangements of regular convex polygons (such as the equilateral triangle, the square, the equilateral pentagon etc.) with equal edge length around a vertex. It is easy to check that there are exactly 17 possible solutions of equation (3.6), and that some of these solutions admit different arrangements of the faces around a vertex, such that there are exactly the following possible vertex types left for Σ after applying EULER's Formula (see also Fig. 2.1.1 and Table 2.1.1 in [25]):

$\rho =$	6	5	4	3
	(3^6)	$(3^3.4^2)$ $(3^2.4.3.4)$ $(3^4.6)$	$(3^2.4.12)$, $(3.4.3.12)$ $(3^2.6^2)$, $(3.6.3.6)$ $(3.4^2.6)$, $(3.4.6.4)$ (4^4)	$(3.7.42)$, $(3.8.24)$ $(3.9.18)$, $(3.10.15)$ (3.12^2), $(4.5.20)$ $(4.6.12)$, (4.8^2) $(5^2.10)$, (6^3)

In particular, it follows that there are no v-transitive, planar strips of degree ≥ 7 and that all faces of a 6-regular, v-transitive, planar strip are triangles.

The fact that a planar graph has a vertex type is even more restrictive. In fact, the following propositions are valid:

Proposition 3.10.
If Γ is a 3-regular, 3-connected, planar graph of the vertex type $(i.j.k)$, then it is impossible that one of the values i, j, and k is odd, and the other two values are distinct. □

Proof:
Consider the proposition to be false. Let v_1 be a vertex and $(i.j.k)$ be the vertex type of Γ, where i is odd and $j \neq k$. Let f be an i-face incident with v_1, and let, in the clockwise order, the other vertices incident with f be denoted by v_2, v_3,..., v_i. Furthermore, let the j-face incident with v_1 be incident with v_2. By turns, it follows that v_2v_3 is a k,i-edge, v_3v_4 is a j,i-edge,..., v_iv_1 a j,i-edge, in spite of the fact that it has to be a k,i-edge. □

It follows that a 3-connected, planar graph cannot have the vertex-types $(3.7.42)$, $(3.8.24)$, $(3.9.18)$, $(3.10.15)$, $(4.5.20)$, and $(5^2.10)$.

Proposition 3.11.
If Γ is a 4-regular, 3-connected, planar graph of the vertex type $(i.j.k.l)$. If one of the numbers i, j, k, and l is odd and its two neighbours are distinct, then there is a triple of pairwise equal numbers among i, j, k, and l. □

Proof:
Consider the statement to be false. Let $(i.j.k.l)$ be a vertex type such that i is odd, $j \neq l$, f be an i-face of Γ being incident with the vertices v_1, v_2,..., v_i of Γ (in the clockwise cyclic order), and the faces around v_1 in the clockwise order be an i-face, a j-face, a k-face, and an l-face. Based on the special role of i, one can distinguish four cases:

(1) $i = j$ and $k = l$,

(2) $k \neq j,l$,

(3) $i \neq j,l$, and

(4) $i \neq j \neq k \neq i$.

It is easy to see the that, by turns, in any of these cases it follows that v_1v_2 is an i,l-edge, v_2v_3 is an i,j-edge, v_3v_4 is an i,l-edge,..., v_1v_2 is an i,j-edge, a contradiction. □

It follows that a 3-connected, planar graph cannot have the vertex types $(3^2.6^2)$, $(3.4^2.6)$, $(3.4.3.12)$, and $(3^2.4.12)$. There remain only the following eleven vertex types:

$\rho =$	6	5	4	3
	(3^6)	$(3^3.4^2)$ $(3^2.4.3.4)$ $(3^4.6)$	$(3.6.3.6)$ $(3.4.6.4)$ (4^4)	(3.12^2) $(4.6.12)$ (4.8^2) (6^3)

These vertex types *can* occur in 3-connected, v-transitive, planar graphs, as can be seen from the well known Archimedean tilings of the plane (e.g. see Fig. 2.1.5 in [25]).

But it is possible to exclude some more of the vertex types which will be done in Theorem 3.13. Its proof relies heavily on Lemma 3.9 which means that, from this point, the v-transitivity and 'striptivity' will have a finger in the pie. The proof of Theorem 3.13 also refers to the following lemma which enables one to recognize κ_∞-separators and λ_∞-edge-separators in v-transitive, planar strips. There is need of such a lemma, since the main parameter to classify the v-transitive, planar strips — apart from the vertex type — is κ_∞, and the parameter λ_∞ is sometimes needed and shall even be given for every classified strip. Mostly, in the subsequent theorems which describe the possible structures of planar, v-transitive strips of a given vertex type there will be constructed a system of disjoint 2-rays fulfilling the conditions (a), (b), and (c) in the first part of the following lemma, and one wants to know whether this set of 2-rays forms a main system.

Lemma 3.12.
Let Σ be a planar strip with the ends $\mathcal{E}$ and $\mathcal{F}$.

(1) Furthermore, let $r \geq 3$ and $\Pi_1, \ldots, \Pi_r$ be a system of disjoint $\mathcal{E},\mathcal{F}$–rays in Σ such that

- *(a) this family spans the vertex set of Σ,*
- *(b) contracting the Π_i, $i \in [r]$, in Σ yields a cycle (with the edges $\Pi_i\Pi_{i+1}$, $i \in [r]$ (indices mod r)), and*
- *(c) there are infinitely many edges between $\Pi_i[w_i,\mathcal{E})$ and $\Pi_{i+1}[w_{i+1},\mathcal{E})$ as well as between $\Pi_i[w_i,\mathcal{F})$ and $\Pi_{i+1}[w_{i+1},\mathcal{F})$ for some vertices $w_i \in V(\Pi_i)$, $w_{i+1} \in V(\Pi_{i+1})$ ($i \in [r]$, indices mod r).*

Then $S = \{v_1, \ldots, v_r\}$ — where $v_i \in V(\Pi_i)$ for all $i \in [r]$ — is an $\mathcal{E},\mathcal{F}$-separator if and only if there exists a sequence $f_1, \ldots, f_r$ of distinct faces of Σ such that, for all $i \in [r]$, v_i is incident with f_{i-1} and f_i but not with any f_j, $j \neq i-1, i$ (indices mod r). Every such separator is a κ_∞-separator of Σ.

(2) Furthermore, let $\Theta_1, \ldots, \Theta_q$ be a system of pairwise edge-disjoint $\mathcal{E},\mathcal{F}$–rays in Σ. Then $T = \{e_1, \ldots, e_q\}$ — where $e_i \in E(\Theta_i)$ for all $i \in [q]$ — is an $\mathcal{E},\mathcal{F}$-edge-separator if there exists a sequence $g_1, \ldots, g_q$ of distinct faces of Σ such that the Θ_i and e_i can be relabelled in such a way that

- *(a) for all $i \in [q]$, e_i is incident with g_{i-1} and g_i but not with any g_j, $j \neq i-1, i$ (indices mod q),*
- *(b) for all $i \in [r]$, there exists $p_i \in [q]$ such that $p_1 < p_2 < \ldots < p_r < p_1$ in some cyclic ordering of $[q]$, and*

(c) for all $p_i < j < j+1 < p_{i+1} - 1$ (indices mod r), e_j and e_{j+1} are consecutive edges between Π_i and Π_{i+1}.

Every such edge-separator is a λ_∞-separator in Σ. □

Proof:
(1),'$\Leftarrow$': Consider $S = \{v_1, ..., v_r\}$ and a sequence f_1,..., f_r of faces of Σ to be as stated in the theorem. Note, that Σ is 3-connected. Let

$$C_1 = \bigcup_{i\in[r]} V(\Pi_i(v_i, \mathcal{E})) \quad \text{and} \quad C_2 = \bigcup_{i\in[r]} V(\Pi_i(v_i, \mathcal{F})).$$

Clearly, C_1 and C_2 give a partition of $V(\Sigma - S)$, and it will be shown that C_1 and C_2 are distinct infinite components of $\Sigma - S$. For that purpose, it suffices to show that there is no edge between a vertex in C_1 and a vertex in C_2.

Assume the contrary, namely that there is an edge xy with $x \in C_1$ and $y \in C_2$. As the following construction will show it follows that Σ contains a subdivision of a $K_{3,3}$, contradicting the planarity of Σ. Let $x \in V(\Pi_i)$ and $y \in V(\Pi_j)$. Clearly, $i \neq j$, since the Π_l are induced. Hence, $j = i-1$ or $j = i+1$ because of (b). Without loss of generality, assume that $j = i-1$ and that there is $k \in [r]$ such that $i < k$. For the following construction, see Fig. 3.3.

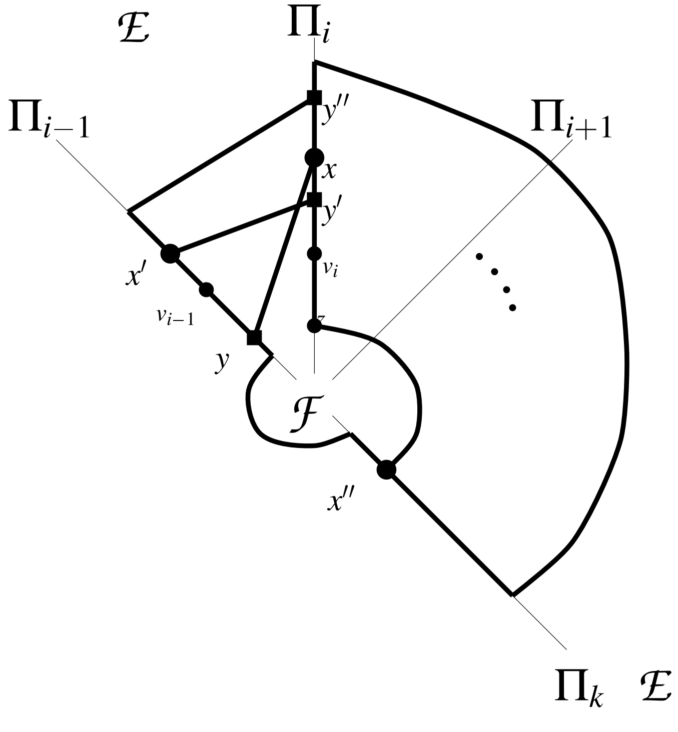


Figure 3.3: The thick lines yield a subdivision of a $K_{3,3}$.

It is impossible that both x and y belong to f_{i-1}. Otherwise, because of Lemma 3.9 (4), some cyclic order of the vertices of f_{i-1} contains the subsequence x, v_i, y, v_{i-1}. Since $r \geq 3$, it is very easy to find a v_i, v_{i-1}-path in Σ which contains neither x nor y. As $xy \in E(\Sigma)$, this contradicts Proposition 3.6.

Hence, without loss of generality, one can assume that x is not incident with f_{i-1}. It follows the existence of a vertex $y' \in V(\Pi_i[v_i,x))$ and of a vertex $x' \in V(\Pi_{i-1}[v_{i-1},\mathcal{E}))$ such that Σ contains the edge $x'y'$ and these edge is incident with f_{i-1}. Furthermore, there exists a vertex $z \in V(\Pi_i[v_i,\mathcal{F}))$ which is incident with f_i such that there is an edge joining z with a vertex from Π_{i+1}. From this vertex, one easily finds a path to some vertex x'' on Π_k which uses only vertices of Π_{i+1},..., Π_k and which has only x'' in common with Π_k. Because of (c) and since Σ is locally finite, one easily finds a vertex $y'' \in V(\Pi_i(x,\mathcal{E}))$ such that there is an edge incident with y'' and a vertex $\in V(\Pi_{i-1}[x',\mathcal{E}))$, and a y'',x''-path which is inner disjoint to all other hitherto constructed paths (go from y'' far enough in direction to $\mathcal{E}$, then go over Π_{i+1}, ..., Π_{k-1} to Π_k, and then go on Π_k in direction to $\mathcal{F}$ till meeting x''). In a very similar way, one finds an x'',y-path which is inner disjoint to all other hitherto constructed paths by first going in direction to $\mathcal{F}$ and then crossing Π_{k+1},..., Π_r, Π_1,..., Π_{i-2}. This completes the construction of a subdivision of a $K_{3,3}$ whose bipartition sets are $\{x,x',x''\}$ and $\{y,y',y''\}$.

(1),'$\Rightarrow$': Let $\{v_1,...,v_r\}$ be an $\mathcal{E},\mathcal{F}$-separating set such that $v_i \in V(\Pi_i)$ for all $i \in [r]$, but assume that v_i and v_{i+1} are not incident with the same face f_i. It follows the existence of an edge between Π_i and Π_{i+1} that crosses the edge v_iv_{i+1} in $\Sigma \cup \{v_iv_{i+1}\}$. From this it is immediate that $\{v_1,...,v_r\}$ cannot be an $\mathcal{E},\mathcal{F}$-separating set, a contradiction.

(2) Consider the 2-rays Θ_1,..., Θ_q, the set $T = \{e_1,...,e_q\}$, the sequence of faces g_1,..., g_q, and p_1,..., p_r to be as stated in the theorem. Clearly, $e_{p_i} \in E(\Pi_i)$ for all $i \in [r]$. It will be shown that T is an $\mathcal{E},\mathcal{F}$–edge-separator. Let D_1 be the set of all vertices belonging to the subray of some $\Theta_i - e_{p_i}$ belonging to $\mathcal{E}$, and let $D_2 = V(\Sigma) - D_1$. It suffices to show that T is an edge-separator of D_1 and D_2.

Assume the contrary. It follows the existence of an edge w_1w_2, where $w_1 \in D_1$ and $w_2 \in D_2$. Without loss of generality, $w_1 \in V(\Pi_i)$ and $w_2 \in V(\Pi_{i-1})$ for some $i \in [r], i > 1$. With the same arguments as above, it follows that it is impossible that both w_1 and w_2 belong to one of the faces $\in \{g_{p_{i-1}}, g_{p_{i-1}+1},...,g_{p_i-1}\} \neq \emptyset$. Now, it can be shown in the same way as above that Σ contains a subdivision of a $K_{3,3}$. □

Theorem 3.13.
There is no v-transitive, planar strip Σ with $m := \kappa_\infty(\Sigma) \geq 3$ which has one of the vertex types (3.12^2), $(3.6.3.6)$, $(4.6.12)$, $(3.4.6.4)$, *and* $(3^4.6)$. □

Proof:
Let $\mathcal{E}$ and $\mathcal{F}$ be the ends of Σ and $\mathcal{S} = \{\Pi_1,...,\Pi_m\}$ be a main system of Σ such that Π_i is adjacent to Π_{i+1} in $\Sigma_\mathcal{S}$ for all $i \in [m]$ (indices mod m). Furthermore, let $V(\Pi_i) := \{v^i_j \mid j \in \mathbb{Z}\}$ and $E(\Pi_i) := \{v^i_j v^i_{j+1} \mid j \in \mathbb{Z}\}$ such that the vertices with positive indices span the subrays of the Π_i that belong to $\mathcal{F}$.

(3.12^2): It can immediately be seen from Lemma 3.9 that if Σ is of degree 3, then it contains no 3-faces, since the 2-rays in $\mathcal{S}$ are induced. Hence, the vertex type (3.12^2) is impossible.

$(3.6.3.6)$: It follows from Lemma 3.9 that every face of Σ has vertices in common exactly with Π_i and Π_{i+1}, for some $i \in [m]$, depending on the face. Now, if Σ has the vertex type $(3.6.3.6)$ then, without loss of generality, there are a vertex $v \in V(\Pi_i)$ and vertices u, $w \in V(\Pi_{i+1})$ such that v, u, and w are the vertices of a 3-face. It follows that the two 6-faces incident with v are also between Π_i and Π_{i+1}, and the only possible arrangement for the remaining 3-face incident with v implies an edge between the neighbours of v on Π_i, which contradicts the fact that Π_i is induced.

(4.6.12): Consider Σ to have the vertex type $(4.6.12)$. Since each vertex is incident with exactly one 4-face, $I := \{V(f) \mid f \text{ is a 4-face}\}$ is an imprimitivity system of Σ, where the $V(f)$ are the blocks of this system. According to Lemma 3.7 and because every translation of Σ induces a translation of Σ_I, Σ_I is a v-transitive, planar strip. Furthermore, it can easily be seen that the edges of each face of Σ are alternatingly incident with faces of the other two covalences.

Consider that there is a 4-face g of Σ such that, for some 12-face or 6-face f of Σ, $V(g) \subset V(f)$. It follows that $V(h) \subset V(f)$ for every 4-face h of Σ that is adjacent to f since Σ is v-transitive and 3-connected. It follows that Σ is either finite or disconnected, a contradiction. It follows that every 6-face (12-face) is adjacent to three (six) distinct 4-faces of Σ, respectively. From this one easily deduces that every 4-face (12-face) is adjacent to two (six) 6-faces of Σ, respectively.

Now, it shall be shown that Σ_I is of degree 4. Consider an arbitrary 4-face h that is adjacent to the distinct 12-faces f and f' and the distinct 6-faces g and g'. Furthermore, denote the 4-face $\neq h$ which is adjacent to g and f (g and f', g' and f, g' and f') with a (b, a', b'), respectively. From the last paragraph it follows that $h \neq a, b, a', b'$ and $a \neq a'$ and $b \neq b'$. Because of symmetry, to show that Σ_I has degree 4 it suffices to prove that $a \neq b'$. If $a = b'$, then the 4,6-edge e which is incident both with a and g is also an edge of b'. e cannot be the edge between b' and g' since $g' \neq g$. It is also impossible that e is the edge between b' and the other 6-face adjacent to b', g'' say, since this would imply $g = g''$ in spite of the fact that both g and g'' are adjacent to f'.

It follows that Σ_I is of degree 4 and that the 12-faces (resp. 6-faces) of Σ become 6-faces (resp. 3-faces) of Σ_I. In fact, Σ_I has the vertex type $(3.6.3.6)$. It has already been shown that this is not possible if $\kappa_\infty(\Sigma_I) \geq 3$. If $\kappa_\infty(\Sigma_I) < 3$, then Σ_I must be one of the strips of degree 4 shown in Fig. 3.1. But Σ_I cannot be the crossed strip since this strip does not contain triangles. It cannot be the alternating crossed ladder, since every edge of Σ_I is contained in a triangle, whereas the alternating crossed ladder contains edges that are not contained in a triangle. And, finally, Σ_I cannot be the two-way infinite antiprism, since the graph spanned by a vertex of Σ_I and its neighbourhood is not isomorphic to the graph spanned by a vertex of the two-way infinite antiprism and its neighbourhood.

(3.4.6.4): It is less easy to show that Σ cannot have the vertex-type $(3.4.6.4)$ (at least with the methods used here). Since each vertex is incident with exactly one 6-face, $I := \{V(f) \mid f \text{ is a 6-face}\}$ is an imprimitivity system of Σ, where the $V(f)$ are the blocks of this system. According to Lemma 3.7, and because every translation of Σ induces a translation of Σ_I, Σ_I is a v-transitive, planar strip. Furthermore, remember that one can speak of "the" faces of Σ.

Consider that Σ contains a 4-face f such that $V(f) \subset V(f')$ for some 6-face f' or there are two 6-faces f and f' such that there are (at least) two 4-faces g and g' between them. In the former case, since Σ is v-transitive, all vertices of f' belong to the same number k of 4-faces whose vertex set belongs completely to f', where either $k = 1$ or $k = 2$. $k = 1$ is impossible since the involved 4-faces are disjoint and $4 \nmid 6$. $k = 2$ implies that Σ is finite or disconnected. Hence, the former case is impossible. In the latter case, note first, that g and

g' cannot have a vertex of f in common, since, otherwise, because of the v-transitivity of Σ, all 4-faces adjacent to f are also adjacent to f', and Σ is finite or disconnected. Hence, because of the v-transitivity of Σ, there are exactly three 4-faces between f and f'. Denote the third one with g''. Now, let, in the clockwise order, the vertices of f (f') be denoted by v_1, v_2, v_3, v_4, v_5, and v_6 (v'_1, v'_2, v'_3, v'_4, v'_5, and v'_6), respectively. Assume, that the vertices of g are, in the clockwise order, v_1, v_6, v'_4, and v'_3. Clearly, Σ contains a vertex v''_5 of a 6-face $f'' \neq f, f'$ such that v''_5, v_1, and v'_3 are the vertices of a 3-face. Then v_2 is adjacent to a vertex v''_4 of f'' such that v_1, v''_5, v''_4, and v_2 are the vertices of a 4-face. Similarly, v'_2 is adjacent to a vertex v''_6 of f''. In a similar way, v'_5, v'_4, v_6 and v_5 are adjacent to vertices of a 6-face f'''. Clearly, $f''' \neq f''$, since, otherwise, because of the v-transitivity of Σ, *all* 4-faces $\neq g, g', g''$ that are adjacent to f and f' are adjacent to the same 6-face f'', and Σ is finite or disconnected. Furthermore, it follows that v'_6, v'_1, v_3, and v_4 are adjacent to the vertices of a 6-face $f'''' \neq f'', f'''$. v_2 is adjacent either to v'_5, v'_6, v'_1, or v'_2. Call this vertex v'_j. Since v_2, v'_j, and v''_4 are the vertices of a 3-face, there exists the edge $v'_j v''_4$ in Σ. Hence, v'_j has a degree of at least 5, a contradiction.

It has been shown that distinct 4-faces that are adjacent to some 6-face f are adjacent to distinct 6-faces $g, h \neq f$. Hence, there is a natural bijection between the 4-faces of Σ and the edges of Σ_I, and the structure of Σ is uniquely determined by Σ_I. Furthermore, Σ_I has the degree six and the vertex type (3^6). $\kappa_\infty(\Sigma_I) < 3$ is impossible, since there exists no such v-transitive, planar strip of the degree 6. Hence, Σ_I is one of the strips determined in Theorem 3.14 with the parameter $p \in \{0, 1, ..., \lfloor \frac{m}{2} \rfloor\}$ (note, that the proof of Theorem 3.14 does not refer to Theorem 3.13).

Let $\mathcal{S} := \{\Pi_1, ..., \Pi_m\}$ be a main system of Σ_I such that $V(\Pi_i) = \{v^i_j \mid j \in \mathbb{Z}\}$ and $E(\Pi_i) = \{v^i_j v^i_{j+1} \mid j \in \mathbb{Z}\}$ for every $i \in [m]$. Furthermore, let Σ_I contain the edges $v^i_j v^{i+1}_j$, $v^i_j v^{i+1}_{j+1}$, $v^m_j v^1_{j-p}$, and $v^m_j v^1_{j+1-p}$ for every $j \in \mathbb{Z}$ and $i \in [m-1]$. For every $i \in [m]$ and $j \in \mathbb{Z}$, v^i_j represents a face of Σ. Let $V(v^i_j) = \{v^i_{j,1}, v^i_{j,2}, ..., v^i_{j,6}\}$ such that $v^i_{j,1}$ ($v^i_{j,2}, v^i_{j,3}, v^i_{j,4}, v^i_{j,5}, v^i_{j,6}$) is both adjacent to some vertex of v^i_{j+1} and of v^{i+1}_{j+1} (v^{i+1}_{j+1} and v^{i+1}_j, v^{i+1}_j and v^i_{j-1}, v^i_{j-1} and v^{i-1}_{j-1}, v^{i-1}_{j-1} and v^{i-1}_j, v^{i-1}_j and v^i_{j+1}), respectively, where the upper indices are taken mod m and one has to appropriately modify some lower indices by p if $i = 1$ or $i = m$ (see Fig. 3.2).

If, for every $i \in [m]$, Π'_i is the 2-ray of Σ induced by the vertices $v^i_{j,4}$, $v^i_{j,5}$, and $v^i_{j,6}$ ($j \in \mathbb{Z}$), and Π''_i is the 2-ray of Σ induced by the vertices $v^i_{j,1}$, $v^i_{j,2}$, and $v^i_{j,3}$ ($j \in \mathbb{Z}$), then $\{\Pi'_1, \Pi''_1, ..., \Pi'_m, \Pi''_m\}$ is a main system of Σ: If $p \neq 0$, then

$$\{v^1_{0,6}, v^1_{0,2}, v^2_{0,6}, v^2_{0,2}, ..., v^{m-p}_{0,6}, v^{m-p}_{0,1}, v^{m-p+1}_{1,5}, v^{m-p+1}_{1,1}, v^{m-p+2}_{2,5}, v^{m-p+2}_{2,1}, ..., v^m_{p,5}, v^m_{p,2}\}$$

separates the ends of Σ, as can be verified easily in the light of Lemma 3.12 (see also Fig. 3.2 for an example with $m = 6$ and $p = 2$). Note, that there is no automorphism of Σ mapping $v^{m-p}_{0,6}$ to $v^{m-p}_{0,5}$, since $v^{m-p}_{0,6} v^{m-p}_{0,1}$ is a 4,6-edge incident with two vertices belonging to a cardinality-minimal end-separating set of Σ, whereas both 4,6-edges incident with $v^{m-p}_{0,5}$ belong to Π'_{m-p} and, therefore, cannot be incident with two vertices belonging to the same cardinality-minimal end-separating set of Σ. Hence Σ is not v-transitive, a contradiction. If $p = 0$, then $S := \{v^1_{1,4}, v^1_{0,1}, v^2_{1,4}, v^2_{0,1}, v^3_{1,4}, v^3_{0,1}, ..., v^m_{0,1}\}$ is a $\kappa_\infty(\Sigma)$-separator such that, for all $i \in [m]$, $v^i_{1,4}$ and $v^i_{0,1}$ are non-adjacent vertices of the same 4-face as well as $v^i_{0,1}$ and $v^{i+1}_{1,4}$ (indices mod m). This contradicts the v-transitivity of Σ, since $v^1_{0,6}$ does not belong to such a $\kappa_\infty(\Sigma)$-separator, because it had also to contain e.g. $v^1_{1,3}$, $v^2_{1,6}$, $v^2_{2,3}$, $v^3_{2,6}$, $v^3_{3,3}$, $v^4_{3,6}$,..., $v^m_{m,3}$, where $v^1_{0,6}$ and $v^m_{m,3}$ do not belong to the same 4-face.

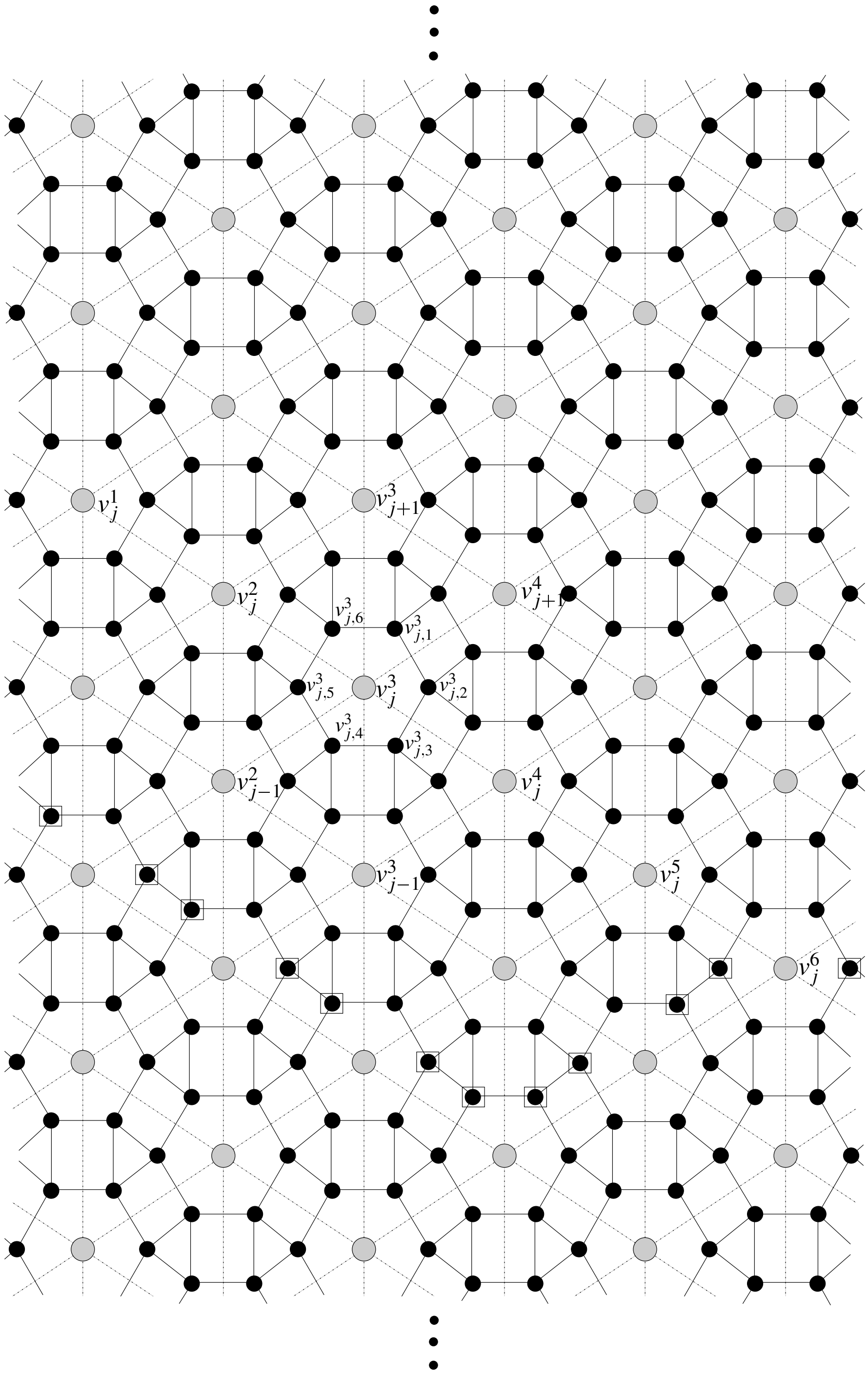


Figure 3.4: The dotted graph represents Σ_I for $m = 6$, whose structure determines the structure of Σ (bold graph). The vertices in the boxes are a κ_∞-separator for $p = 2$.

$(3^4.6)$: Now, let Σ have the vertex type $(3^4.6)$. Then, at each vertex v, in the cyclic ordering of the edges around v, there are a 3,6-edge, three 3,3-edges and a 3,6-edge. The 3,3-edge, which is between the other two 3,3-edges (called the *central edge of* v), cannot be the central edge of its other endvertex u. Otherwise, for every $v \in V(\Sigma)$, the central edge of v is also the central edge of its other endvertex, v' say, and there is a common neighbour w of v' and v such that wv' and wv are 3,3-edges. It follows, that neither $v'w$ nor vw is a central edge of w, a contradiction to the fact that $v'w$ and vw are consecutive (in the cyclic ordering of the edges around w).

It follows that, for every $v \in V(\Sigma)$, its central edge vx and one of its non-central 3,3-edges vu belong to the same edge orbit. The other non-central 3,3-edge vw of v does not belong to this edge orbit, since, otherwise, uv is the central edge of u and wv is the central edge of w. But then ux and wx are also 3,3-edges, and xv is the central edge of both x and v, which has already been disproved above. It follows, that vw belongs to an edge orbit O that is a 1-factor of Σ.

Clearly, $\Sigma - O$ is planar. $\Sigma - O$ is also connected and has the vertex-type $(3.4.6.4)$: Let, in the clockwise cyclic order, the neighbours of some vertex v be u, w, x, y, and z (in Σ), where, without loss of generality, $vw \in O$ and u and z belong to the same 6-face as v. yv is the central edge of y in Σ. Hence, xy is a 3,3-edge of Σ. It follows, that xw cannot be a 3,3-edge, since otherwise xv is the central edge of both x and v. Therefore, xy is the central edge of x and wu the central edge of w. It also follows that $yz \in O$.

Since $\Sigma - O$ is connected and $Aut(\Sigma - O) \supseteq Aut(\Sigma)$, $\Sigma - O$ is a v-transitive strip. As has been shown above, there exist no v-transitive, planar strips with $\kappa_\infty \geq 3$ and the vertex type $(3.4.6.4)$. Hence, $\Sigma - O$ has connectivity either 1 or 2. Since $\Sigma - O$ has degree 4, it must be some of the strips of connectivity 2 and degree 4 shown in Fig. 3.1. $\Sigma - O$ cannot be the crossed strip, since this strip contains no triangles. $\Sigma - O$ cannot be the alternating crossed ladder, since every vertex of this strip is contained in exactly one induced 4-cycle and three 4-cycles that induce the same K_4 (and in no other 4-cycle), whereas, in $\Sigma - O$, each vertex is contained in at least two 4-cycles that do not induce a K_4. Finally, $\Sigma - O$ cannot be the 2-way infinite antiprism, since it has an edge orbit that spans a single ray, whereas each edge orbits of $\Sigma - O$ spans infinitely many finite components. □

3.2.1 $\rho = 6$

If Σ is a 3-connected planar graph of even degree, then two edges e and e' at some vertex v are called *opposite edges (at v)*, if, in the clockwise cyclic ordering of the edges around v, there are as many edges between e and e' as are edges between e' and e.

Theorem 3.14.
Σ is a planar strip of the vertex type (3^6) with $m := \kappa_\infty(\Sigma) \geq 3$ if and only if Σ has one of the following structures (see Figs. 3.5, 3.6, 3.7):

$$\begin{aligned} V(\Sigma) &= \{v_j^i \mid i \in [m], j \in \mathbb{Z}\}, \\ E(\Sigma) &= \{v_j^i v_{j+1}^i \mid i \in [m], j \in \mathbb{Z}\} \cup \\ &\quad \{v_j^i v_j^{i+1}, v_j^i v_{j+1}^{i+1} \mid i \in [m-1], j \in \mathbb{Z}\} \cup \\ &\quad \{v_j^m v_{j-p}^1, v_j^m v_{j+1-p}^1 \mid j \in \mathbb{Z}\}, \end{aligned}$$

where p ranges over 0, 1,..., $\lfloor \frac{m}{2} \rfloor$. Furthermore, if Σ has one of the above structures, then the following statements are valid:

(1) Σ is v-transitive.

(2) $\lambda_\infty(\Sigma) = 2\kappa_\infty(\Sigma)$.

(3) The structure of Σ is uniquely determined by κ_∞ and p.

(4) There exist three edge-disjoint 2-factors Σ_a, Σ_b, and Σ_c of Σ such that

(a) each of these 2-factors uses opposite edges at every vertex of Σ,

(b) the edge sets of these 2-factors together form a partition of $E(\Sigma)$,

(c) the components of each of these 2-factors are either exclusively 2-rays or exclusively cycles of length m,

(d) at least one of these 2-factors consists of exactly m 2-rays, which is also the sum of the numbers of 2-rays contained in the other two 2-factors implying that there is at most one 2-factor containing cycles,

(e) the edges of each Σ_x, $x \in \{a,b,c\}$, are contained in a single edge orbit of Σ, and

(f) the edges of distinct Σ_x, Σ_a and Σ_b say, are in the same edge orbit if and only if Σ_a and Σ_b have an equal number of components — which is the case if and only if Σ_c contains cycles or if $\kappa_\infty(\Sigma)$ is even and Σ_a as well as Σ_b has $\frac{\kappa_\infty(\Sigma)}{2}$ many components. □

Proof:
Let Σ be a planar strip of the vertex type (3^6) with $m \geq 3$. According to Lemma 3.9 (3), Σ is 3-connected and has a unique embedding $\mathfrak{S}$ in the sense of Lemma 3.4. Note that two edges uv and uw are consecutive (in the cyclic ordering of the edges around u) if and only if they are edges of the same 3-face of Σ.

If one starts at an arbitrary edge and, inductively, generates an edge sequence by adding step by step the next two uniquely determined opposite edges, then one obtains a connected subgraph Π_1^a of Σ of degree 2, thus a cycle or a 2-ray. Otherwise, Π_1^a contains an edge

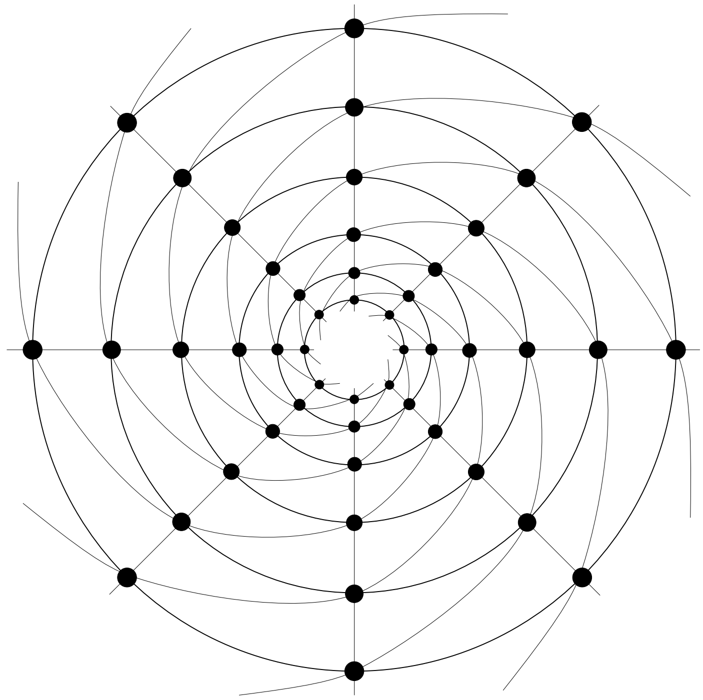


Figure 3.5: A plane embedding in a center-cylindric perspective of the uniquely determined planar strip of the vertex type (3^6) with $\kappa_\infty = 8$ and $p = 0$.

sequence $v_1^1v_2^1$, $v_2^1v_3^1$,..., $v_k^1v_1^1$, where the v_i^1 are distinct vertices, such that $v_1^1v_2^1$ and $v_k^1v_1^1$ are either consecutive or there is exactly one edge between them (in the cyclic ordering of the edges around v_1^1). Assume k to be chosen smallest possible. In the former case, as Σ has the vertex type (3^6) and because of the construction of Π_1^a, one easily obtains the existence of vertices w_1, w_2,..., w_{k-3} and edges $v_2^1v_k^1$, $v_2^1w_1$, $v_3^1w_1$, $v_3^1w_2$, $v_4^1w_2$,..., $v_{k-1}^1w_{k-3}$, $v_{k-1}^1w_1$, $v_k^1w_1$, and w_1w_2, w_2w_3,..., w_kw_1. (For instance, v_1^1, v_2^1 and v_k^1 induce a 3-face, implying the existence of the edge $v_2^1v_k^1$. As $v_1^1v_2^1$ and $v_2^1v_3^1$ are opposite edges, there is an edge v_2^1x between $v_2^1v_3^1$ and $v_2^1v_k^1$ (in the cyclic ordering of the edges around v_2^1). Similarly, there is an edge v_k^1y between $v_k^1v_2^1$ and $v_k^1v_{k-1}^1$ (in the cyclic ordering of the edges around v_k^1). Obviously, $v_3^1 \neq x = y \neq v_{k-1}^1$, and there is an edge v_3^1x, because all faces of Σ are 3-faces. As v_3^1x, v_2^1x, and v_k^1x are consecutive edges (in the cyclic ordering of the edges around x), x cannot be one of the v_i^1, $4 \leq i \leq k-2$. Rename x as w_1.) Note that $k \geq 6$, since otherwise one gets a contradiction fast. The edges between the w_i again form a sequence of consecutive opposite edges, but a shorter one. This contradicts the minimality of k. In the latter case (there is exactly one edge between $v_1^1v_2^1$ and $v_k^1v_1^1$), a similar argumentation is feasible.

Case (I): Π_1^a is a 2-ray.

Assume that there have been constructed n 2-rays Π_1^a,..., Π_n^a with the vertex sets $V(\Pi_i^a) = \{v_j^i \mid j \in \mathbb{Z}\}$ and the edge sets $E(\Pi_i^a) = \{v_j^iv_{j+1}^i \mid j \in \mathbb{Z}\}$ ($i \in [n]$) such that there are the additional edges $v_j^iv_j^{i+1}$ and $v_j^iv_{j+1}^{i+1}$ for all $j \in \mathbb{Z}$, $i \in [n-1]$. First assume that $V(\Sigma)$ is

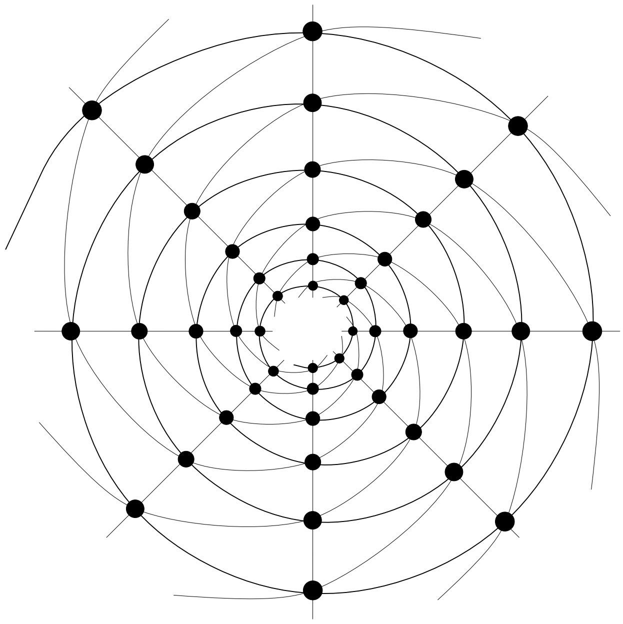


Figure 3.6: A plane embedding in a center-cylindric perspective of the uniquely determined planar strip of the vertex type (3^6) with $\kappa_\infty = 8$ and $p = 1$.

spanned by the Π_i^a, $i \in [n]$, and that there is an edge $v_j^1 v_k^1$ such that $j < k$ and $|j-k| > 1$ is smallest possible. Then $v_k^1 v_j^1$ and $v_k^1 v_{k-1}^1$ are not consecutive (in the cyclic ordering of the edges around v_k^1) because otherwise $|j-k|$ is not smallest possible. Hence the edges $v_k^1 v_{k+1}^1$ and $v_k^1 v_j^1$ are consecutive (in the cyclic ordering of the edges around v_k^1), and there exists the edge $v_j^1 v_{k+1}^1$ which is consecutive with the edges $v_j^1 v_k^1$ and $v_j^1 v_{j+1}^1$ (in the cyclic ordering of the edges around v_j^1), since $v_j^1 v_k^1$ is also not consecutive with $v_j^1 v_{j+1}^1$ (in the cyclic ordering of the edges around v_j^1). It follows the existence of the edge $v_{j+1}^1 v_{k+1}^1$ which cannot be consecutive with the edge $v_{k+1}^1 v_k^1$ (in the cyclic ordering of the edges around v_{k+1}^1) and, therefore, is consecutive with the edge $v_{k+1}^1 v_{k+2}^1$ (in the cyclic ordering of the edges around v_{k+1}^1) etc. It is easy to see that, inductively, one obtains the existence of an edge $v_{k+1}^1 v_{2k-j+1}^1$ which is possible only if $n = 1$. In this case, for $p = j - k$, one obtains exactly the edges $v_j^1 v_{j-p}^1$ and $v_j^1 v_{j-p+1}^1$, $j \in \mathbb{Z}$.

It has been shown that if $n > 1$ and the Π_i^a, $i \in [n]$, span $V(\Sigma)$, then Π_1^a (and symmetrically, of course, also Π_n^a) are induced 2-rays. It is easy to see that, with the techniques used so far, for some $p \in \mathbb{Z}$, one obtains the missing edges $v_i^1 v_{i+p}^n$ and $v_i^1 v_{i+p-1}^n$ ($i \in \mathbb{Z}$). (It is impossible that the edges are of the form $v_i^1 v_{p-i}^n$, since otherwise Σ has only one end.) To finish the induction, it remains to show that if the Π_i^a, $i \in [n]$, do not span $V(\Sigma)$, then there is a 2-ray Π_{n+1}^a with the vertex set $V(\Pi_{n+1}^a) = \{v_j^{n+1} \mid j \in \mathbb{Z}\}$ and the edge set $E(\Pi_{n+1}^a) = \{v_j^{n+1} v_{j+1}^{n+1} \mid j \in \mathbb{Z}\}$ such that there are the additional edges $v_j^n v_j^{n+1}$ and $v_j^n v_{j+1}^{n+1}$ ($j \in \mathbb{Z}$).

Without loss of generality, there exists a vertex v_i^{n+1} which is adjacent to v_i^n such that $v_i^n v_i^{n+1}$ is consecutive with $v_i^n v_{i-1}^n$ (in the cyclic ordering of the edges around v_i^n). It immediately follows the existence of the edge $v_i^{n+1} v_{i-1}^n$. Let $e = v_i^{n+1} x \neq v_{i+1}^n v_{i-1}^n$ be the other edge

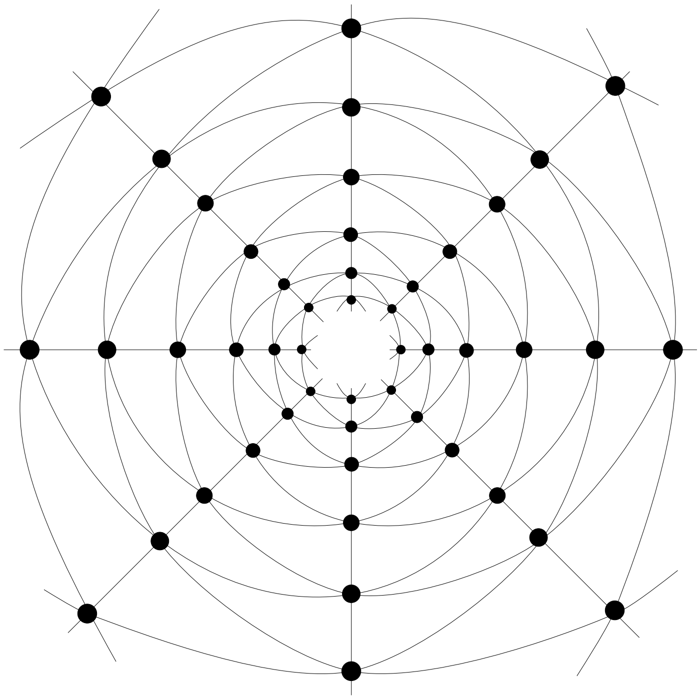


Figure 3.7: A plane embedding in a center-cylindric perspective of the uniquely determined planar strip of the vertex type (3^6) with one system containing $\kappa_\infty = 8$ many pairwise disjoint 2-rays and two systems containing $\frac{\kappa_\infty}{2}$ many pairwise disjoint 2-rays, respectively, which is not possible if κ_∞ is odd.

which is consecutive with $v_i^{n+1}v_i^n$ (in the cyclic ordering of the edges around v_i^{n+1}). Then x is also adjacent to v_i^n and v_{i+1}^n. Hence $x \notin V(\Pi_1^a)$ and $x \neq v_j^n$ for all $j \in \mathbb{Z}$, $j \neq i-1, i, i+1, i+2$, since Σ is 6-regular. Clearly, $x \neq v_j^n$ for $j \in \{i-1, i, i+1, i+2\}$. It follows that x is a new vertex. Set $v_{i+1}^{n+1} := x$. With similar arguments, one shows the existence of vertices v_{i-2}^{n+1} and v_{i+1}^{n+1} that are incident with the two opposite edges of e, respectively. Inductively, one constructs a 2-ray Π_{n+1}^a as desired above, using the additional argument that none of the edges in the sequence of opposite edges starting with e is incident with an already constructed vertex v_j^{n+1}, since either this implies that the edges of the sequence are the edges of a cycle or this contradicts the fact that Σ has the degree 6. In the former case, one also gets a vertex of degree > 6.

Case (II): Π_1^a is a cycle.

With very similar arguments and methods as those used in (I) one yields that Π_1^a is an *induced* cycle, with the vertices $v_1^1,\ldots, v_k^1$ say (in this cyclic order), there are vertices $v_1^0,\ldots, v_k^0$ and $v_1^2,\ldots, v_k^2$ and edges $v_j^1v_j^0$, $v_j^1v_{j+1}^0$, $v_j^1v_j^2$, $v_j^1v_{j+1}^2$, $v_j^0v_{j+1}^0$, $v_j^2v_{j+1}^2$ ($j \in [k]$, indices mod k) such that $v_1^0,\ldots, v_k^0$ ($v_1^2,\ldots, v_k^2$) induce two disjoint cycles Π_2^a (Π_0^a) of length k, respectively, which together span the neighbourhood of Π_1^a. Inductively, one obtains a system $(\Pi_i^a)_{i\in\mathbb{Z}}$ of cycles of length k such that, for each $i \in \mathbb{Z}$, $V(\Pi_i^a) = \{v_1^i,\ldots, v_k^i\}$ and Σ contains the edges $v_j^iv_j^{i+1}$, $v_j^iv_{j+1}^{i+1}$ ($j \in [k]$, indices mod k). As every $V(\Pi_i^a)$, $i \in \mathbb{Z}$, separates the ends of Σ, and as Σ_b and Σ_c — as postulated in (4) — are easy to find and both have the cardinality $k = |V(\Pi_0^a)|$, it follows $k = m$. Because Σ_b and Σ_c are edge-disjoint, the edges between two cycles Π_i^a and Π_{i+1}^a are separating the ends of Σ, the number of those edges is $2\kappa_\infty$, and the statements about the edge orbits and the v-transitivity of Σ are obvious now, the case where one of the systems Σ_a, Σ_b, and Σ_c contains cycles is completely settled.

Now, let Σ_a be a 2-factor with infinite components as constructed in (I). Let $|\Sigma_a| = n$, and adopt the number p given above. The reader can easily deduce the existence of 2-factors Σ_b, Σ_c having exactly $|n-p|$, $|p|$ infinite components, respectively, such that (4)(a), (b), and (c) are settled. If $p = 0$, then Σ_c contains cycles, and the structure of Σ is as shown in (II). Otherwise, if $p < 0$ then $n - p = n + |p|$. If $0 < p \leq n$ then $n = |p| + |n-p|$. If $p > n$, then $p = n + |n-p|$, which settles part of (4)(d). It remains to show that there is one system of cardinality $\kappa_\infty(\Sigma)$, and the assertion concerning $\lambda_\infty(\Sigma)$. Note that all 2-rays constructed here have subrays in both ends of Σ. Otherwise, if this is not true for a 2-ray contained in some Σ_x ($x \in \{a,b,c\}$), then this is not true for all 2-rays in Σ_x which contradicts the fact that Σ_x spans $V(\Sigma)$, since Σ_x is finite.

Without loss of generality, assume $0 < p < n$, and $p > n - p$. Then, according to Lemma 3.12, the following sets of vertices (edges) separate the ends of Σ, respectively:

$$\{v_0^1, v_1^2, \ldots, v_p^{p+1}, v_p^{p+2}, \ldots, v_p^n\}$$

$$(\{v_0^1 v_1^1\ ,\ v_1^1 v_1^2\ ,\ v_1^2 v_2^2\ ,\ v_2^2 v_2^3\ ,\ \ldots\ ,\ v_p^p v_p^{p+1}\ ,\ v_p^{p+1} v_{p+1}^{p+1}\ ,\ v_p^{p+1} v_{p+1}^{p+2}\ ,\ v_p^{p+2} v_{p+1}^{p+2}\ ,$$

$$v_p^{p+2} v_{p+1}^{p+3}\ ,\ \ldots\ ,\ v_p^n v_{p+1}^n\ ,\ v_p^n v_1^1\}).$$

In the other cases, one easily finds similar cardinality-minimal end-separating sets yielding $m = max(n, |n-p|, |p|)$. This settles (4)(d) and (2). If $p < 0$ ($p > n$) then one could have started in (I) with a 2-ray Π_1^a belonging to a system of $n - p > n$ ($p > n$) pairwise v-disjoint 2-rays of Σ, respectively. Then $0 \leq p \leq n$ would have been valid. If so, then $m = n$ and one gets all possible structures of Σ by choosing $p \in \{0, 1, \ldots, \lfloor \frac{m}{2} \rfloor\}$, since p and $n - p$ play symmetric roles, which settles also (3). It is not difficult to see that the statements about the edge orbits are correct and that all constructed graphs are v-transitive.

If Σ has one of the structures as described in the theorem, then it is immediate that Σ is a planar strip of the vertex type (3^6) with $m \geq 3$. □

Remark 3.15.
The proof of Theorem 3.14 almost shows that the only infinite, connected planar graphs of the vertex type (3^6) are those listed in the theorem and the one given by the tiling of the Euclidean plane with equilateral triangles of equal size. In the proof, there is a passage where it is argued that if $\Pi_1^a, \ldots, \Pi_n^a$ span $V(\Sigma)$, then Π_1^a and Π_n^a cannot be glued together by edges of the form $v_i^1 v_{p-i}^n$, since otherwise Σ has only one end. This passage has to be replaced by a passage that shows that Σ contains a subdivision of a $K_{3,3}$ if Π_1^a and Π_n^a are glued together in the wrong direction. □

3.2.2 $\rho = 5$

The Vertex Type $(3^3.4^2)$

Theorem 3.16.
Σ is a planar strip of the vertex type $(3^3.4^2)$ with $m := \kappa_\infty(\Sigma) \geq 3$ if and only if the following statements are valid:

(1) The 2-factor Σ_1 of Σ which contains exactly all 3,4-edges of Σ consists either of infinitely many disjoint induced cycles Ξ_i, $i \in \mathbb{Z}$, all of the same length l, or of k disjoint induced 2-rays $\Pi_1,\ldots, \Pi_k$, where $k \in \mathbb{N}$ is even.

(2) If Σ_1 consists of cycles, then $\kappa_\infty(\Sigma) = \lambda_\infty(\Sigma) = l$. Furthermore, the structure of Σ is uniquely determined by l and can be described as follows (see Fig. 3.8):

$$\begin{aligned} V(\Sigma) &= \{v_j^i \mid i \in \mathbb{Z}, j \in [l]\} \\ E(\Sigma) &= \{v_j^i v_{j+1}^i \mid i \in \mathbb{Z}, j \in [l], \textit{ lower indices mod } l\} \cup \\ &\quad\ \{v_j^i v_j^{i+1} \mid i \in \mathbb{Z}, j \in [l]\} \cup \\ &\quad\ \{v_j^i v_{j+1}^{i+1} \mid i \in \mathbb{Z}, i \textit{ even}, j \in [l], \textit{ lower indices mod } l\}. \end{aligned}$$

(3) If Σ_1 consists of 2-rays, then Σ has one of the following structures (see Fig. 3.9):

$$\begin{aligned} V(\Sigma) &= \{v_j^i \mid i \in [k], j \in \mathbb{Z}\}, \\ E(\Sigma) &= \{v_j^i v_{j+1}^i \mid i \in [k], j \in \mathbb{Z}\} \cup \\ &\quad\ \{v_j^i v_j^{i+1} \mid i \in [k-1], j \in \mathbb{Z}\} \cup \\ &\quad\ \{v_j^k v_{j+p}^1 \mid j \in \mathbb{Z}\} \\ &\quad\ \{v_j^i v_{j+1}^{i+1} \mid i \in [k], 4 \mid (i-1), j \in \mathbb{Z}\} \cup \\ &\quad\ \{v_j^i v_{j-1}^{i+1} \mid i \in [k], 4 \mid (i-3), j \in \mathbb{Z}\}, \end{aligned}$$

where $p \in \mathbb{N}_0$. The structure of Σ is determined by the parameters k and p. Furthermore, $\kappa_\infty(\Sigma) = max\{k, p + \lceil \frac{k}{4} \rceil\}$ and $\lambda_\infty(\Sigma) = max\{\frac{3}{2}k, k + p + \lceil \frac{k}{4} \rceil\}$.

If Σ is a planar strip of the vertex type $(3^3.4^2)$ with $\kappa_\infty(\Sigma) \geq 3$, then Σ is v-transitive. □

Proof:
Let Σ be a planar strip of the vertex type $(3^3.4^2)$ with $m \geq 3$. As Σ is 3-connected (Lemma 3.9 (3)), it makes sense to speak of *the* faces etc. of Σ (Lemma 3.4). The subgraph Σ_1 of Σ which contains exactly the 3,4-edges of Σ is a 2-factor of Σ, thus each of its components is a cycle or a 2-ray.

First, assume that Σ_1 has at least one component Ξ_0 which is a cycle. Let $V(\Xi_0) = \{v_j^0 \mid j \in [l]\}$ and $E(\Xi_0) = \{v_j^0 v_{j+1}^0 \mid j \in [l]$, indices mod $k\}$. Consider the 4,4-edge $v_1^0 u$. If $u = v_j^0$ for some $j \in [l]$ then it is easy to see that l is even, $j = \frac{l}{2} + 1$, and $v_2^0 v_{j+1}^0, \ldots, v_{j-1}^0 v_l^0$ are 4,4-edges of Σ. Now, consider the 3,3-edge $v_1^0 w$ of Σ which is consecutive with $v_1^0 v_l^0$ (in the cyclic ordering of the edges around v_1^0). Obviously, $w \notin V(\Xi_0)$, since $w \in V(\Xi_0)$ implies that

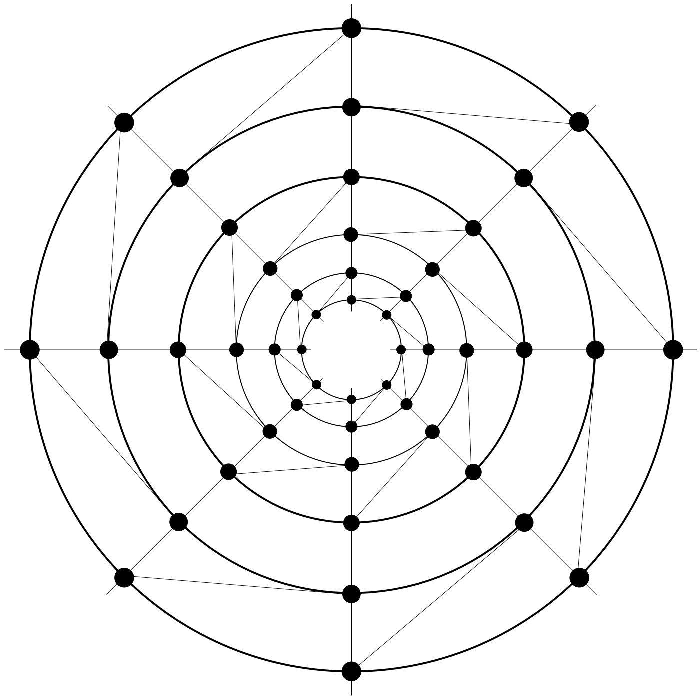


Figure 3.8: A plane embedding in a center-cylindric perspective of the uniquely determined planar strip of the vertex type $(3^3.4^2)$, where the orbit of the 3,4-edges spans infinitely many cycles of length $l = \kappa_\infty = 8$.

Σ is finite or disconnected. Set $v_1^1 := w$. Clearly, $v_l^0 v_1^1$ is a 3,3-edge of Σ which is consecutive with a 3,4-edge $v_1^1 v_l^1$ (in the cyclic ordering of the edges around v_1^1). Furthermore, $v_1^0 v_1^1$ is consecutive with a 3,4-edge $v_1^1 v_2^1$ (in the cyclic ordering of the edges around v_1^1). It follows the existence of the 3,3-edges $v_1^0 v_2^1$, which is consecutive with $v_1^0 v_2^0$ (in the cyclic ordering of the edges around v_1^0) and $v_1^1 v_2^1$ (in the cyclic ordering of the edges around v_2^1), and $v_2^1 v_2^0$, which is consecutive with $v_1^0 v_2^0$ (in the cyclic ordering of the edges around v_2^0) and a 3,4-edge $v_2^1 v_3^1$ (in the cyclic ordering of the edges around v_2^1). Inductively, one obtains a cycle Ξ_1 with $V(\Xi_1) = \{v_j^1 \mid j \in [l]\}$ and $E(\Xi_1) = \{v_j^1 v_{j+1}^1 \mid j \in [l]$, indices mod $l\}$ such that, for each $j \in [l]$, there are the 3,3-edges $v_j^0 v_j^1$ and $v_j^0 v_{j+1}^1$ in Σ (indices mod l). It is impossible that there is a 4,4-edge incident with two vertices of Ξ_1, since this would imply that Σ is finite or disconnected. Inductively, one obtains a structure of Σ as described in the theorem, apart from the fact that Σ has only one end, a contradiction. Hence, $u \notin V(\Xi_0)$. It follows the existence of a cycle Ξ_{-1} with $V(\Xi_{-1}) = \{v_j^{-1} \mid j \in [l]\}$ and $E(\Xi_{-1}) = \{v_j^{-1} v_{j+1}^{-1} \mid j \in [l]$, indices mod $l\}$ such that, for each $j \in [l]$, there is the 4,4-edge $v_j^0 v_j^{-1}$ in Σ. The existence of a 3,3-edge $v_j^0 v_i^{-1}$ implies that Σ is finite or disconnected. Hence, it is easy to see now that Σ has a structure as described in the theorem. Furthermore, it is easy to see that Σ is v-transitive and, with the aid of Lemma 3.12, that $\lambda_\infty(\Sigma) = \kappa_\infty(\Sigma) = l$. Note, that the structure of Σ is uniquely determined by $\kappa_\infty(\Sigma)$.

Now, consider that all components of Σ_1 are 2-rays. Since Σ is a strip, there are only

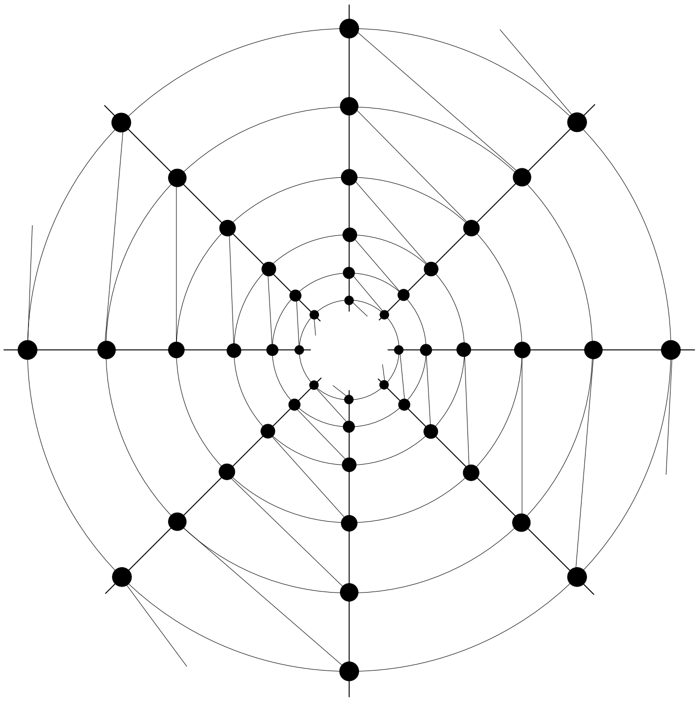


Figure 3.9: A plane embedding in a center-cylindric perspective of the uniquely determined planar strip of the vertex type $(3^3.4^2)$, where $p = 0$ and the orbit of the 3,4-edges spans $k = \kappa_\infty = 8$ disjoint 2-rays.

finitely many such 2-rays, $\Pi_1,\ldots,\Pi_k$ say. For $i \in [k]$, let $V(\Pi_i) = \{v^i_j \mid j \in \mathbb{Z}\}$ and $E(\Pi_i) = \{v^i_j v^i_{j+1} \mid j \in \mathbb{Z}\}$.

1. The Π_j, $j \in \mathbb{Z}$, are induced.

Assume that there is a 3,3-edge $v^i_r v^i_s$ $(r < s)$, and consider $s-r$ to be minimal. The other 3,3-edge incident with v^i_r is incident with some neighbour v of v^i_s, since the two 3,3-edges incident with any vertex z of Σ are consecutive (in the cyclic ordering of the edges around z). If vv^i_s is a 3,3-edge, then there is a vertex w such that wv^i_s is a 3,4-edge which is consecutive with vv^i_s (in the cyclic ordering of the edges around v^i_s) and wv is a 3,4-edge which is consecutive with vv^i_s (in the cyclic ordering of the edges around v). Since wv and wv^i_s are incident with the same 3-face of Σ, they are consecutive (in the cyclic ordering of the edges around w), which contradicts the cyclic ordering of the edges around w. Furthermore, vv^i_s cannot be a 4,4-edge since v, v^i_r, and v^i_s are the vertices of a 3-face of Σ. It follows that $v = v^i_{s-1}$ or $v = v^i_{s+1}$. The former case is impossible since $s-r$ was chosen minimally. Similar, one shows that there is a 3,3-edge incident with v^i_s and v^i_{r-1}. There is an edge incident with v^i_{s+1} and v^i_{r+1} since the 3,4-edge $v^i_r v^i_{r+1}$ and the 3,3-edge $v^i_r v^i_{s+1}$ are consecutive (in the cyclic ordering of the edges around v^i_r), and this edge is a 3,3-edge because it cannot be a 4,4-edge and there are no 3,4-edges left. Now, it is possible to show the existence of the 3,3-edges $v^i_{r+1}v^i_{s+2}$, $v^i_{s+2}v^i_{r+2},\ldots,$ $v^i_s v^i_{s+(s-r)}$ inductively, a contradiction to the fact that there are only two 3,3-edges incident with v_s^i. In a similar manner, one can show that there are no 4,4-edges connecting vertices of the same Π_j. This finishes the proof of 1.

With similar arguments as used above, one can easily show that the Π_i and their vertices can be relabelled in such a way that, for some $p \in \mathbb{N}_0$, Σ has a structure as described in the theorem (note that Π_1 and Π_k cannot be glued together in the reverse direction, since Σ has two ends). Because of symmetry, a negative integer value for p yields a strip isomorphic to the strip given by $-p(\in \mathbb{N}_0)$. In particular, k is even. That Σ is v-transitive can be seen from the following arguments:

a) For every $s \in \mathbb{Z}$, let $\varphi_s : V(\Sigma) \to V(\Sigma)$ be a map such that, for all $i \in [k]$ and for all $j \in \mathbb{Z}$, $\varphi(v_j^i) = v_{j+s}^i$. Then $\varphi_s \in Aut(\Sigma)$. It follows that the vertices of each Π_i, $i \in [k]$, belong to a single vertex orbit of Σ, respectively.

b) For even $r \in [k]$, let $\varphi^r : V(\Sigma) \to V(\Sigma)$ be a map such that, for all $i \in [k]$, $j \in \mathbb{Z}$,

$$\varphi^r(v_j^i) = \begin{cases} v_j^{i+r} \quad , & i+r \leq k \\ v_{j+p}^{i+r-k}, & i+r > k \end{cases} .$$

Then $\varphi^r \in Aut(\Sigma)$. It follows that all the vertices of all Π_i with even (odd) index belong to a single edge orbit, respectively.

c) If $4 \nmid k$, then let $\psi : V(\Sigma) \to V(\Sigma)$ be a map such that, for all $i \in [k]$ and $j \in \mathbb{Z}$, $\psi(v_j^i) = v_{-j}^{k-i+1}$. Then $\psi \in Aut(\Sigma)$. In particular, $\psi(v_{j+p}^1 v_j^k) = v_{-j-p}^k v_{-j}^1$. If $4 \mid k$, then let $\psi' : V(\Sigma) \to V(\Sigma)$ be a map such that, for all $i \in [k]$ and $j \in \mathbb{Z}$,

$$\psi'(v_j^i) = \begin{cases} v_{-j}^{k-i+1}, & 4|i,(i+1) \\ v_{1-j}^{k-i+1}, & 4|(i+2),(i+3) \end{cases} .$$

Then $\psi' \in Aut(\Sigma)$. In particular, $\psi'(v_{j+p}^1 v_j^k) = v_{-j-p}^k v_{-j}^1$. Since both ψ and ψ' map a vertex from a Π_i with even index to a Π_i with odd index, Σ has only one vertex orbit.

2. $\kappa_\infty(\Sigma) = \max\{k, p + \lceil \frac{k}{4} \rceil\}$

First, assume that $p + \lceil \frac{k}{4} \rceil \leq k$. If $p > 0$, then

$$S = \{v_p^1, v_p^2, v_{p-1}^3, v_{p-2}^4, v_{p-3}^5, v_{p-3}^6, v_{p-4}^7, v_{p-5}^8, v_{p-6}^9, v_{p-6}^{10}, v_{p-7}^{11}, \ldots, \\ v_1^{p+\lceil \frac{p}{4} \rceil}, v_1^{p+\lceil \frac{p}{4} \rceil+1}, \ldots, v_1^k\}$$

separates the ends $\mathcal{E}$ and $\mathcal{F}$ of Σ (Lemma 3.12). If $p = 0$, then, clearly, $S' = \{v_0^1, \ldots, v_0^k\}$ separates the ends $\mathcal{E}$ and $\mathcal{F}$ of Σ. Note that $|S| = |S'| = k$. It follows 2., since $\Pi_1, \ldots, \Pi_k$ is a system of k pairwise disjoint rays in Σ.

Now, assume that $p + \lceil \frac{k}{4} \rceil > k$. Let $q = k - \lceil \frac{k}{4} \rceil$ and $r = p + \lceil \frac{k}{4} \rceil - k$. Then

$$S = \{v_{q+r}^1, v_{q+r-1}^1, \ldots, v_{q+1}^1, v_q^1, v_q^2, v_{q-1}^3, v_{q-2}^4, v_{q-3}^5, v_{q-3}^6, \\ v_{q-4}^7, v_{q-5}^8, v_{q-6}^9, v_{q-6}^{10}, v_{q-7}^{11}, \ldots, v_1^k\}$$

separates the ends $\mathcal{E}$ and $\mathcal{F}$ of Σ (Lemma 3.12), since it is not difficult to find a system of $p+\lceil\frac{k}{4}\rceil$ many pairwise disjoint rays in Σ such that these rays together span Σ and each of these rays has in common exactly one vertex with S. Note that $|S| = r+k = \lceil\frac{k}{4}\rceil$.

3. $\lambda_\infty(\Sigma) = \max\{\frac{3}{2}k,\ k+p+\lceil\frac{k}{4}\rceil\}$.
Note that it is very easy to find systems of $\frac{3}{2}k$ $(k+p+\lceil\frac{k}{4}\rceil)$ many pairwise edge-disjoint 2-rays in Σ, respectively. If $\frac{3}{2}k \geq k+p+\lceil\frac{k}{4}\rceil$, then let S contain the edges

$$v^1_p v^1_{p-1}\,,\ v^1_{p-1}v^2_p\,,\ v^2_p v^2_{p-1}\,,\ v^3_p v^3_{p-1}\,,\ v^3_{p-1}v^4_{p-1}\,,\ v^4_{p-1}v^4_{p-2}\,,\ v^5_{p-1}v^5_{p-2}\,,$$
$$v^5_{p-2}v^6_{p-1}\,,\ v^6_{p-1}v^6_{p-2}\,,\ v^7_{p-1}v^7_{p-2}\,,\ v^7_{p-2}v^8_{p-2}\,,\ \ldots\,,\ v^s_1 v^s_0$$

such that s is smallest possible in this sequence, and, for every $i \in \{s,\ldots,k\}$, let S contain the additional edges $v^i_1 v^i_0$ and (if $4 \mid (i-1)$) $v^i_0 v^{i+1}_1$ and (if $4 \mid (i-3)$) $v^i_1 v^{i+1}_0$ (indices mod k). Clearly, S contains $\frac{3}{2}k$ edges. If $\frac{3}{2}k < k+p+\lceil\frac{k}{4}\rceil$, then let $q = p-\lfloor\frac{k}{4}\rfloor$ $(= k+p+\lceil\frac{k}{4}\rceil-\frac{3}{2}k)$ and let S contain the edges

$$v^1_p v^k_0\,,\ v^1_{p-1}v^k_{-1}\,,\ v^1_{p-2}v^k_{-2}\,,\ \ldots\,,\ v^1_{p-q+1}v^k_{-q+1}\,,\ v^1_{p-q+1}v^1_{p-q}\,,\ v^1_{p-q}v^2_{p-q+1}\,,$$
$$v^2_{p-q+1}v^2_{p-q}\,,\ v^3_{p-q+1}v^3_{p-q}\,,\ v^3_{p-q}v^4_{p-q}\,,\ v^4_{p-q}v^4_{p-q-1}\,,\ v^5_{p-q}v^5_{p-q-1}\,,\ v^5_{p-q-1}v^6_{p-q}\,,$$
$$v^6_{p-q}v^6_{p-q-1}\,,\ v^7_{p-q}v^7_{p-q-1}\,,\ v^7_{p-q-1}v^8_{p-q-1}\,,\ \ldots\,,\ v^k_1 v^k_0.$$

Note, that S contains $q+\frac{3}{2}k = k+p+\lceil\frac{k}{4}\rceil$ many edges. Either case, Lemma 3.12 yields 3.

If Σ has one of the structures given in the theorem then it is immediate that Σ is a planar strip of the vertex type $(3^3.4^2)$ with $m \geq 3$. $\square$

The Vertex Type $(3^2.4.3.4)$

Theorem 3.17.
Σ is a planar strip of the vertex type $(3^2.4.3.4)$ with $m := \kappa_\infty(\Sigma) \geq 3$ if and only if m is even and Σ has one of the following structures (see Fig. 3.10 for the case $p = 0$):

$$\begin{aligned}
V(\Sigma) &= \{v^i_j \mid i \in [m], j \in \mathbb{Z}\},\\
E(\Sigma) &= \{v^i_j v^i_{j+1} \mid i \in [m], j \in \mathbb{Z}\} \cup\\
&\quad\ \{v^i_j v^{i+1}_j \mid i \in [m-1], j \in \mathbb{Z}\} \cup\\
&\quad\ \{v^m_j v^1_{j+p} \mid j \in \mathbb{Z}\} \cup\\
&\quad\ \{v^i_{2j} v^{i+1}_{2j-1} \mid i \in [m], i \text{ odd}, j \in \mathbb{Z},\ \text{upper indices mod } m\} \cup\\
&\quad\ \{v^i_{2j} v^{i+1}_{2j+1} \mid i \in [m-2], i \text{ even}, j \in \mathbb{Z}\} \cup\\
&\quad\ \{v^m_{2j} v^1_{2j+1+p} \mid j \in \mathbb{Z}\},
\end{aligned}$$

where p ranges over 0, 2,..., m. Furthermore, if Σ has one of the above structures, then the following statements are valid:

(1) Σ *is v-transitive if and only if* $p = 0$.

(2) $\lambda_\infty(\Sigma) = \frac{3m+p}{2}$.

(3) Σ *is uniquely determined by the parameters m and p.*

(4) There exist two 2-factors Σ_a *and* Σ_b *of* Σ *such that these two systems together contain exactly all 3,4-edges of* Σ, Σ_a *contains m 2-rays (and nothing else), and* Σ_b *contains either infinitely many cycles of length m (and nothing else) or an even number p of 2-rays, where* $0 < p \leq m$ *(and nothing else).* □

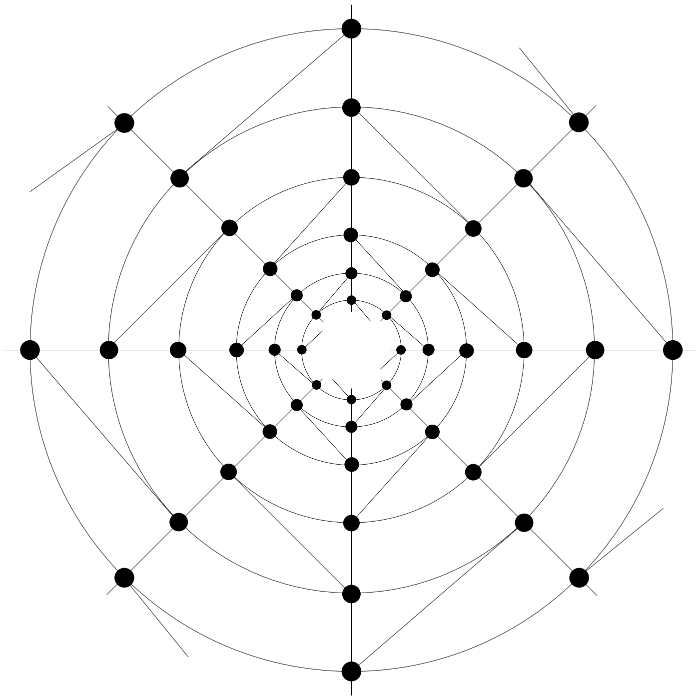


Figure 3.10: A plane embedding in a center-cylindric perspective of the uniquely determined planar strip of the vertex type $(3^2.4.3.4)$ with $\kappa_\infty = 8$ and $p = 0$.

Proof:
If Σ is a planar strip of the vertex type $(3^2.4.3.4)$ with $m \geq 3$, then each vertex of Σ is incident with exactly one 3,3-edge. Hence the set of all 3,3-edges of Σ represents a 1-factor of Σ. Deleting all these edges from Σ, one obtains a planar strip Σ_1 of the vertex type (4^4). It follows that if $\kappa_\infty(\Sigma_1) \geq 3$, then Σ_1 is one of the strips described in Theorem 3.18. (Note that Theorem 3.18 refers neither to this theorem nor its proof.)

Consider $\kappa_\infty(\Sigma_1) \geq 3$ and let Σ_a, Σ_b, m, and p be as described in Theorem 3.18 (with respect to Σ_1). If Σ_b contains a cycle of odd length, then Σ_a contains an odd number of 2-rays. Let the 2-rays in Σ_a be denoted by Π_1,..., Π_m, with the vertex set $V(\Pi_i) = \{v^i_j \mid j \in \mathbb{Z}\}$, respectively, such that the vertices on each Π_i are indexed in a natural order, and let all v^i_j, $j \in [m]$, are on the same 2-ray/cycle of Σ_b. Further, let Π_i be adjacent to Π_{i+1} (indices mod m), and let m be odd. Without loss of generality, assume that there is a 3,3-edge between

v_1^1 and v_2^2 in Σ. Then there is also a 3,3-edge between v_3^2 and v_2^3, since Σ has the vertex type $(3^2.4.3.4)$. It follows that there are the 3,3-edges

$$v_1^3 v_2^4,\ v_3^4 v_2^5,\ v_1^5 v_2^6, \ldots,\ v_1^m v_{2+p}^1,\ v_{3+p}^1 v_{2+p}^2,$$

where p is the number of 2-rays in Σ_b. The latter edge is impossible, since there are only 3,3-edges of the form $v_j^1 v_{j+1}^2$ between Π_1 and Π_2, as can be verified easily. This proves that m must be even. With a similar argumentation one shows that p is even, too.

Σ_1 is f-transitive. It follows that, for each choice of even numbers m ($m \geq 3$) and p ($0 \leq p \leq m$), there are at most two planar strips of the vertex type $(3^2.4.3.4)$, since the position of one 3,3-edge determines the position of all other 3,3-edges of Σ and a 4-face admits two possibilities for inserting a 3,3-edge. It is easy to see that if Σ has one of the structures described in the theorem (which cover one half of the possibilities), then mapping v_j^i to v_{j+1}^{i+1} for all $i \in [m-1]$ and $j \in \mathbb{Z}$ and v_j^m to v_{j+p+1}^1 for all $j \in \mathbb{Z}$ yields an isomorphism between Σ and the graph obtained from Σ by switching all 3,3-edges in their 4-face simultaneously. Hence, Σ has one of the structures described in the theorem and cannot have any other structure.

Clearly, if Σ has one of the structures described in the theorem, then Σ is a planar strip of the vertex type $(3^2.4.3.4)$, and Lemma 3.12 yields that

$$\{v_{p-1}^1,\ v_{p-2}^2,\ v_{p-3}^3, \ldots,\ v_0^p,\ v_0^{p+1}, \ldots,\ v_0^m\}$$

separates the ends of Σ. It follows that $\kappa_\infty(\Sigma) = m$. Furthermore, one gets $\frac{3m+p}{2}$ edge-disjoint 2-rays in Σ by taking the m 2-rays from Σ_a and, for every even $i \in [m+p]$, adding the 2-ray defined by the vertex sequence

$$\ldots,\ v_{i+m+p}^1,\ v_{i+m+p-1}^2,\ v_{i+m+p-1}^1,\ v_{i+m-2}^m,\ v_{i+m-2}^{m-1},\ v_{i+m-3}^m,\ v_{i+m-3}^{m-1},\ v_{i+m-4}^{m-2},$$
$$v_{i+m-4}^{m-3},\ v_{i+m-5}^{m-2},\ v_{i+m-5}^{m-3},\ v_{i+m-6}^{m-4},\ \ldots,\ v_i^2,\ v_i^1,\ v_{i-1}^2,\ v_{i-1}^1,\ v_{i-p-2}^m,\ v_{i-p-2}^{m-1},$$
$$v_{i-p-3}^m,\ v_{i-p-3}^{m-1},\ v_{i-p-4}^{m-2},\ \ldots\ .$$

Lemma 3.12 yields that

$$\{v_p^1 v_{p-1}^1,\ v_{p-1}^1 v_{p-1}^2,\ v_{p-2}^1 v_{p-2}^2,\ v_{p-2}^2 v_{p-3}^2,\ v_{p-2}^3 v_{p-3}^3,\ v_{p-3}^3 v_{p-3}^4,\ v_{p-4}^3 v_{p-4}^4,$$
$$v_{p-4}^4 v_{p-5}^4,\ v_{p-4}^5 v_{p-5}^5, \ldots,\ v_2^{p-1} v_1^{p-1},\ v_1^{p-1} v_1^p,\ v_0^{p-1} v_0^p,\ v_0^p v_{-1}^p,\ v_0^p v_{-1}^{p+1},\ v_0^{p+1} v_{-1}^{p+1},$$
$$v_0^{p+2} v_{-1}^{p+2},\ v_0^{p+2} v_{-1}^{p+3},\ v_0^{p+3} v_{-1}^{p+3},\ \ldots,\ v_0^{m-1} v_{-1}^{m-1},\ v_0^{m-1} v_{-1}^m,\ v_0^m v_{-1}^m\}$$

separates $\mathcal{E}$ and $\mathcal{F}$ in Σ and contains $\frac{3m+p}{2}$ edges. Hence, $\lambda_\infty(\Sigma) = \frac{3m+p}{2}$. Finally, it is easy to see that Σ is v-transitive if $p = 0$, and that Σ is not v-transitive if $0 \neq p \neq m$. The reader may verify this by mapping some vertex v_j^i of Σ to v_{j+1}^i and using the fact that this determines uniquely the positions where all the other vertices are mapped to if the map is an isomorphism. One gets quickly a contradiction. If $p = m$ then Σ is not v-transitive since $\{v_0^1, v_1^2, v_2^3, \ldots, v_{m-1}^m\}$ separates the ends of Σ and v_0^2 is not contained in such an end-separator of Σ (starting at v_0^2 and, by turns, adding the non-adjacent vertex of the current 4-face does not yield a cycle of faces).

Now, assume that $\kappa_\infty(\Sigma_1) < 3$. Then the proof of Theorem 3.18 yields that there are exactly two possible structures for Σ_1, namely the structures described there for $r = 1$, $q = 2$ and $r = 2$, $q = 2$. Clearly, $r = 1$ is impossible. If $r = 2$, then the structure of Σ is uniquely determined (can be seen in the same way as above), and, according to Lemma 3.12, $\kappa_\infty(\Sigma) = 2$. □

3.2.3 $\rho = 4$

Theorem 3.18.
Σ is a planar strip of the vertex type (4^4) with $m := \kappa_\infty(\Sigma) \geq 3$ if and only if Σ has one of the following structures (see Figs. 3.11, 3.12, 3.13):

$$\begin{aligned} V(\Sigma) &= \{v_j^i \mid i \in [m], j \in \mathbb{Z}\}, \\ E(\Sigma) &= \{v_j^i v_{j+1}^i \mid i \in [m], j \in \mathbb{Z}\} \cup \\ &\quad \{v_j^i v_j^{i+1} \mid i \in [m-1], j \in \mathbb{Z}\} \cup \\ &\quad \{v_j^m v_{j+p}^1 \mid j \in \mathbb{Z}\}, \end{aligned}$$

where p ranges over 0, 1,..., m. Furthermore, if Σ has one of the above structures, then the following statements are valid:

(1) Σ is v-transitive, self-dual, and f-transitive. Σ is e-transitive if and only if $p = m$.

(2) $\lambda_\infty(\Sigma) = m + p$.

(3) The structure of Σ is uniquely determined by the parameters κ_∞ and p.

(4) There exist two edge-disjoint 2-factors Σ_a and Σ_b of Σ such that

  (1) each 2-factor uses opposite edges at every vertex of Σ,
  (2) the edge sets of these 2-factors together form a partition of $E(\Sigma)$,
  (3) Σ_a contains exactly m 2-rays (and nothing else) and Σ_b contains p 2-rays and, if $p = 0$, infinitely many cycles of length m (and nothing else), and
  (4) the edges of Σ_a and Σ_b are contained in a single edge orbit, respectively. □

Proof:
Let Σ be a planar strip of the vertex type (4^4) with $\kappa_\infty(\Sigma) \geq 3$. According to Lemma 3.9 (3), Σ is 3-connected and has a unique embedding $\mathfrak{S}$ in the sense of Lemma 3.4.

Let $v \in V(\Sigma)$. If one starts with two opposite edges at v and, by turns, adds the next two edges which are opposite to the edges which are outermost in the sequence so far, one yields either a cycle or a 2-ray. (Otherwise, Σ contains a 'loop', with the vertex set $\{v_1, v_2,..., v_k\}$ and the edges v_1v_2, v_2v_3,..., v_kv_1 say, such that, for all $i \in [k-1]$, v_iv_{i+1} and $v_{i+1}v_{i+2}$ (indices mod k) are opposite edges at v_{i+1}, and v_1v_2, v_kv_1 are consecutive edges (in the cyclic ordering of the edges around v_1). Assume k to be smallest possible. (It is easy to see that $k \geq 4$.) v_1, v_2 and v_k belong to the same 4-face f_1 of Σ. Let w_1 denote the fourth vertex of this face. Clearly, $w_1 \neq v_i$ $(i = 4,...,k-2)$ since v_2w_1 and v_kw_1 are consecutive (in the cyclic ordering of the edges around w_1), and the two vacant edges at v_i $(i = 4,...,k-2)$ are not consecutive. Furthermore, it is easy to see that $w_1 \neq v_3, v_{k-1}$. v_3, v_2, and w_1 belong to the same 4-face f_2 of Σ. Denote the fourth vertex of this face by w_2. With the same arguments, one shows that w_2 cannot be an already constructed vertex. By turns, one can construct vertices w_3, w_4,..., w_{k-2} such that Σ contains the edges w_iw_{i+1} $(i \in [k-2]$, indices mod $k-2)$, i.e. Σ contains a 'loop' of length $k-2$, contradicting the minimality of k.)

First, consider that one yields a cycle Ξ_0 with $V(\Xi_0) = \{v_0^i \mid i \in [m]\}$ and $E(\Xi_0) = \{v_0^i v_0^{i+1} \mid i \in [m]$, upper indices mod $m\}$ such that m is smallest possible. Ξ_0 is induced, since otherwise there is an edge $v_0^i v_0^j$ such that $q = j - i > 1$ is smallest possible, implying

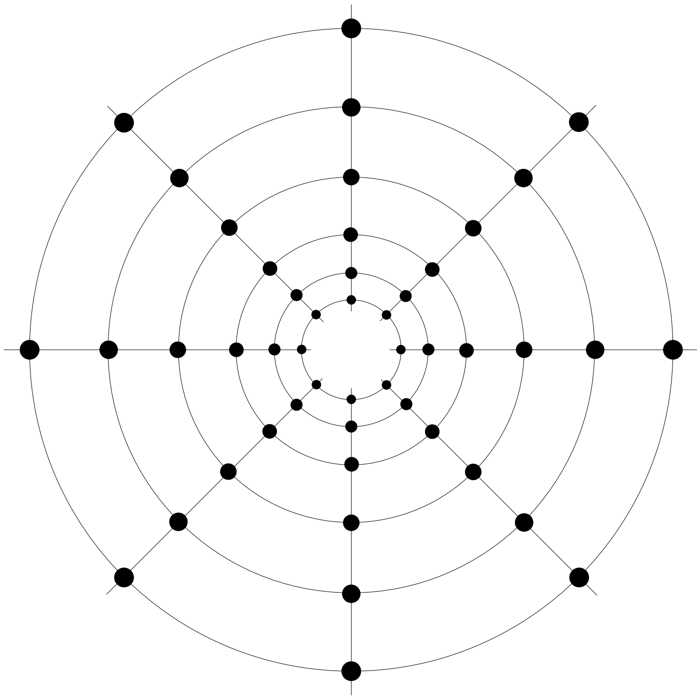


Figure 3.11: A plane embedding in a center-cylindric perspective of the uniquely determined planar strip of the vertex type (4^4) with the parameters $\kappa_\infty = 8$ and $\lambda_\infty = 8$. One edge orbit spans eight disjoint 2-rays, and the second edge orbit spans infinitely many disjoint cycles.

the existence of either $v_0^{i+1}v_0^{j+1}$ (contradicting Proposition 3.6, as $v_0^{i+1}v_0^{j+1}$ and $v_0^iv_0^j$ belong to the same 4-face of Σ and Σ contains planar subgraphs containing $v_0^{i+1}v_0^{j+1}$ and $v_0^iv_0^j$ where Ξ_0 is a facial cycle) or $v_0^{i+1}v_0^{j-1}$ (contradicting the minimality of q, since $(j-1)-(i+1) \leq 1$ is also impossible). For all $i \in [m]$, let $v_1^i \notin V(\Xi_0)$ be a vertex adjacent to v_0^i. Clearly, Σ contains the edges $v_1^iv_1^{i+1}$ ($i \in [m]$, lower indices mod m). Furthermore, all the v_1^i, $i \in [m]$, are distinct, since otherwise one yields either an opposite-edge-cycle of length $< m$ (contradicting the minimality of m) or a 'loop' (which has already been excluded). There is no edge $v_0^iv_1^j$ with $i \neq j$, because this would imply that Σ is either finite or disconnected. Now, it is easy to see that, inductively, one can construct cycles Ξ_j, $j \in \mathbb{Z}$, with $V(\Xi_j) = \{v_j^i \mid i \in [m]\}$ and $E(\Xi_j) = \{v_j^iv_j^{i+1} \mid i \in [m]$, upper indices mod $m\}$ such that, for every $i \in [m]$ and $j \in \mathbb{Z}$, Σ contains the additional edge $v_j^iv_{j+1}^i$, yielding the structure given by the theorem for $p = 0$. With the aid of Lemma 3.12, it is not difficult to see that $\kappa_\infty(\Sigma) = \lambda_\infty(\Sigma) = m$. Furthermore, it is very easy to see that all other statements are valid.

Now, consider that one yields a 2-ray Π_1 with $V(\Pi_1) = \{v_j^1 \mid j \in \mathbb{Z}\}$ and $E(\Pi_1) = \{v_j^1v_{j+1}^1 \mid j \in \mathbb{Z}\}$. If $V(\Pi_1) = V(\Sigma)$, then it is easy to show that, without loss of generality, Σ contains the additional edges $v_j^1v_{j+q}^1$ for some $q \in \mathbb{N} \setminus \{1\}$. Otherwise, there exists a vertex v_j^2 which is adjacent to some vertex of Π_1, v_j^1 say, and it is not difficult to show that Π_1 is induced and that the sequence of opposite edges passing through v_j^2 which does not contain the edge $v_j^1v_j^2$ yields a 2-ray Π_2 with $V(\Pi_2) = \{v_j^2 \mid j \in \mathbb{Z}\}$ and $E(\Pi_2) = \{v_j^2v_{j+1}^2 \mid j \in \mathbb{Z}\}$

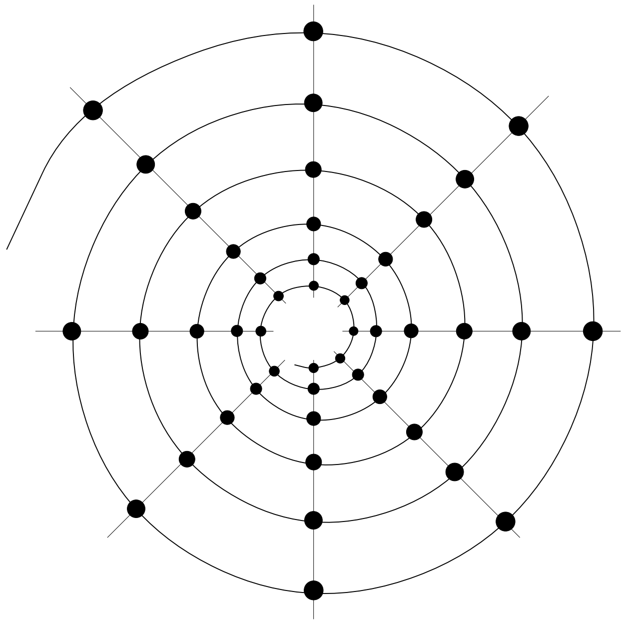


Figure 3.12: A plane embedding in a center-cylindric perspective of the uniquely determined planar strip of the vertex type (4^4) with the parameters $\kappa_\infty = 8$ and $\lambda_\infty = 9$. The second edge orbit here spans one 2-ray. The greater λ_∞ the more similar the second edge orbit becomes to the first one.

such that, for every $j \in \mathbb{Z}$, there exists the additional edge $v_j^1 v_j^2$. If Π_1 and Π_2 together span Σ, then it is easy to show that, without loss of generality, Σ contains the additional edges $v_j^2 v_{j+q}^1$ for some $q \in \mathbb{N} \setminus \{1\}$. Otherwise, one can show by induction that, for some $3 \leq r \in \mathbb{N}$, there are r 2-rays $\Pi_1,..., \Pi_r$ such that

$$\begin{aligned} V(\Sigma) &= \{v_j^i \mid i \in [r], j \in \mathbb{Z}\}, \\ E(\Sigma) &= \{v_j^i v_{j+1}^i \mid i \in [r], j \in \mathbb{Z}\} \cup \\ &\quad \{v_j^i v_j^{i+1} \mid i \in [r-1], j \in \mathbb{Z}\} \cup \\ &\quad \{v_j^p v_{j+q}^1 \mid j \in \mathbb{Z}\} \end{aligned}$$

for some $q \in \mathbb{N}$ ($q = 0$ yields a structure which has already been determined in the cycle case). With the help of Lemma 3.12 one can easily show that $m = \kappa_\infty(\Sigma) = max\{q,r\}$ and, provided that $max\{q,r\} \geq 3$, Σ can be relabelled such that Σ has one of the structures in the theorem ($p = min\{q,r\}$). Furthermore, Lemma 3.12 yields that $\lambda_\infty(\Sigma) = m + p$.

Clearly, if Σ has one of the structures given in the theorem, then Σ is a planar strip of the vertex type (4^4) with $m \geq 3$ and the statements (1)–(4) are valid. □

3.2.4 $\rho = 3$

If Σ is a 3-regular, v-transitive, planar strip with and $\kappa_\infty(\Sigma) \geq 3$, then $\lambda_\infty(\Sigma) = \kappa_\infty(\Sigma)$, since $\lambda_\infty(\Sigma) \geq \kappa_\infty(\Sigma)$ and any pair of edge-disjoint rays in Σ is also vertex-disjoint.

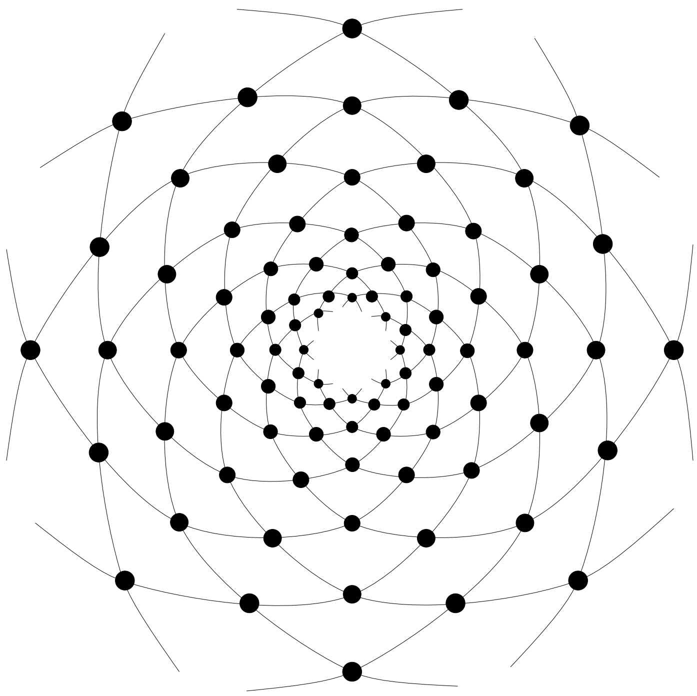


Figure 3.13: A plane embedding in a center-cylindric perspective of the uniquely determined planar strip of the vertex type (4^4) with the parameters $\kappa_\infty = 8$ and $\lambda_\infty = 16$. Note, that this graph is e-transitive.

The Vertex Type (6^3)

Theorem 3.19.
Σ is a planar strip of the vertex type (6^3) with $m := \kappa_\infty(\Sigma) \geq 3$ if and only if Σ has one of the following structures (see Figs. 3.14, 3.15, 3.16):

$$\begin{array}{rcl} V(\Sigma) & = & \{v^i_j \mid i \in [m], j \in \mathbb{Z}\}, \\ E(\Sigma) & = & \{v^i_j v^i_{j+1} \mid i \in [m], j \in \mathbb{Z}\} \cup \\ & & \{v^i_j v^{i+1}_{j+1} \mid i \in [m-1], j \in \{..., -3, -1, 1, 3, ...\}\} \cup \\ & & \{v^m_j v^1_{j+1-p} \mid j \in \{..., -3, -1, 1, 3, ...\}\}, \end{array}$$

where p ranges over 0, 2, 4,..., $2\lfloor \frac{m}{2} \rfloor$. Furthermore, if Σ has one of the above structures, then Σ is v-transitive and is uniquely determined by the parameters m and p, and $\lambda_\infty(\Sigma) = \kappa_\infty(\Sigma)$.
□

Proof:
The dual Γ^* of a planar, connected graph Γ of the vertex type (6^3) is a planar, connected graph of the vertex type (3^6). Since the intersection of the boundary of two 6-faces of Σ either is empty or contains at least one edge, and Σ contains at least three pairwise disjoint 2-rays, it is easy to find a main system of Σ^* of cardinality at least three, i.e. $\kappa_\infty(\Sigma^*) \geq 3$.

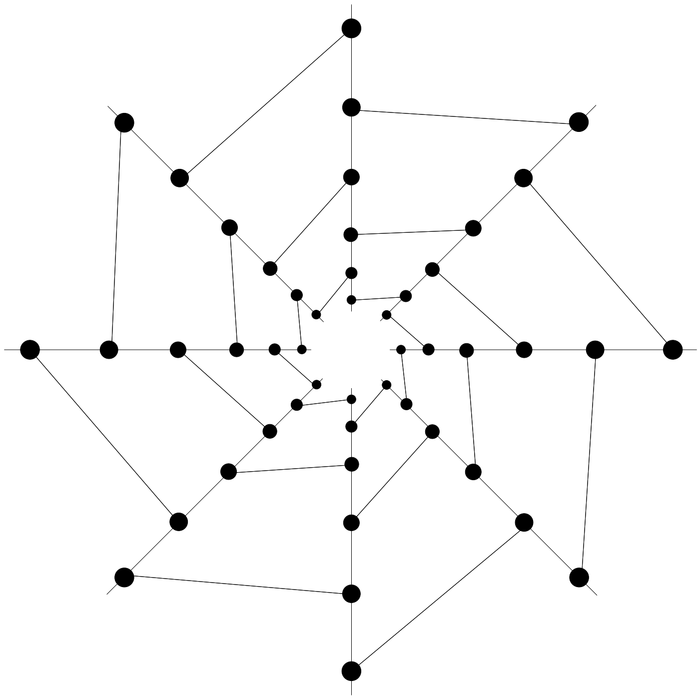


Figure 3.14: A plane embedding in a center-cylindric perspective of the uniquely determined planar strip of the vertex type (6^3) with the parameters $\kappa_\infty = 8$ and $p = 0$.

Hence Σ^* is one of the graphs determined in Theorem 3.14. It follows, that Σ is one of the graphs as specified in the theorem, since Σ^* has also a unique dual, namely Σ.

Now, if Σ is one of the graphs as specified in the theorem, then it clearly is a planar strip with the vertex type (6^3) with $\kappa_\infty(\Sigma) \geq 3$. Furthermore, Σ is v-transitive. With the aid of Lemma 3.12, it is easy to show that $\kappa_\infty(\Sigma) = m$. That $\lambda_\infty(\Sigma) = \kappa_\infty(\Sigma)$ has been mentioned at the beginning of this section. □

The Vertex Type (4.8^2)

Theorem 3.20.
Γ is a planar graph of the vertex type (4.8^2) with $m := \kappa_\infty(\Sigma) \geq 3$ if and only if m is even and Σ has the following structure (see Fig. 3.17):

$$\begin{aligned} V(\Sigma) &= \{v_j^i \mid i \in [m], j \in \mathbb{Z}\}, \\ E(\Sigma) &= \{v_j^i v_{j+1}^i \mid i \in [m], j \in \mathbb{Z}\} \cup \\ &\quad \{v_j^i v_j^{i+1} \mid i \in [m], j \in \mathbb{Z}, (2 \nmid i \wedge (4 \mid j \vee 4 \mid (j+1))) \vee \\ &\quad (2 \mid i \wedge (4 \mid (j+2) \vee 4 \mid (j+3)))\}. \end{aligned}$$

If Σ has one of the above structures, then $\kappa_\infty(\Sigma) = \lambda_\infty(\Sigma)$ and Σ is v-transitive. □

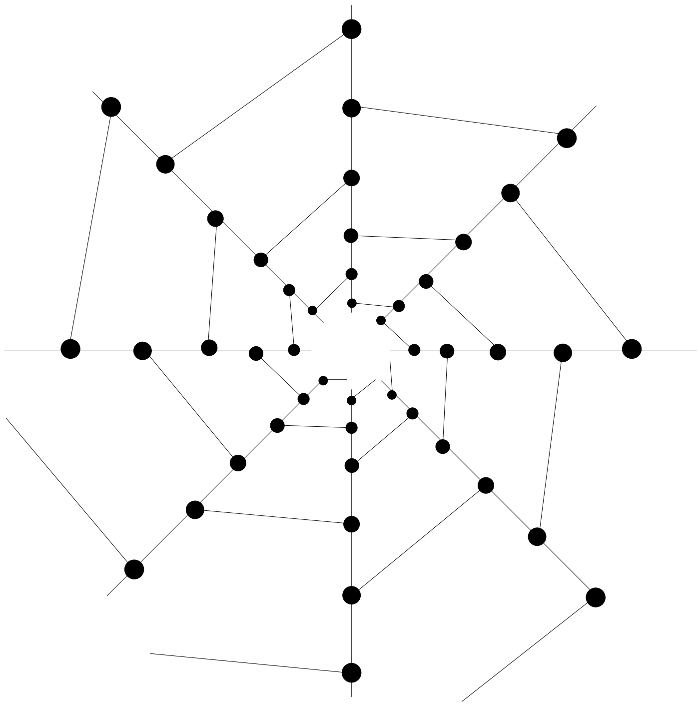


Figure 3.15: A plane embedding in a center-cylindric perspective of the uniquely determined planar strip of the vertex type (6^3) with the parameters $\kappa_\infty = 8$ and $p = 2$.

Proof:
Let Σ be a planar strip of the vertex type (4.8^2) with $m \geq 3$. According to Lemma 3.9 (3), Σ is 3-connected and has a unique embedding $\mathfrak{S}$ in the sense of Lemma 3.4. As each vertex is incident with exactly one 4-face, $I := \{V(f) \mid f \text{ is a 4-face}\}$ is an imprimitivity system of Σ, where the $V(f)$ are the blocks of this system. Due to Proposition 3.7 and the fact that there exists a translation of Σ which induces a translation of Σ_I, Σ_I is a planar strip.

Consider f to be a 4-face of Σ, $V(f) = \{u,v,w,x\}$ say, and uv to be an 8,8-edge of Σ. Then wx is also an 8,8-edge of Σ, since Σ is v-transitive. This contradicts the fact that Σ is connected and infinite.

Now, consider two 4-faces f, f' of Σ and two edges uu', vv' such that $u,v \in V(f)$ and $u',v' \in f'$.

If u and v as well as u' and v' are not adjacent, then the vertices $\neq u,v$ of f, w and x say, are adjacent to non-adjacent vertices of some 4-face f'' of Σ, since Σ is v-transitive. If $f'' = f'$, then Σ is either finite or disconnected. If $f'' \neq f'$, then one can find the next 4-face f''' of Σ in the same manner. By induction, it is easy to see that Σ is either finite or disconnected or (if one gets a two-way infinite sequence of 4-faces) has connectivity 2.

If u and v as well as u' and v' are adjacent then, because of the v-transitivity of Σ, either at least one of the vertices $\neq u,v$ of f is adjacent to some vertex of f' (and, hence, Σ is not 3-connected), or the vertices $\neq u,v$ of f are adjacent to adjacent vertices of some 4-face $f'' \neq f'$ of Σ. Inductively, one gets a two-way infinite sequence of 4-faces in such a way that Σ has connectivity 2, a contradiction.

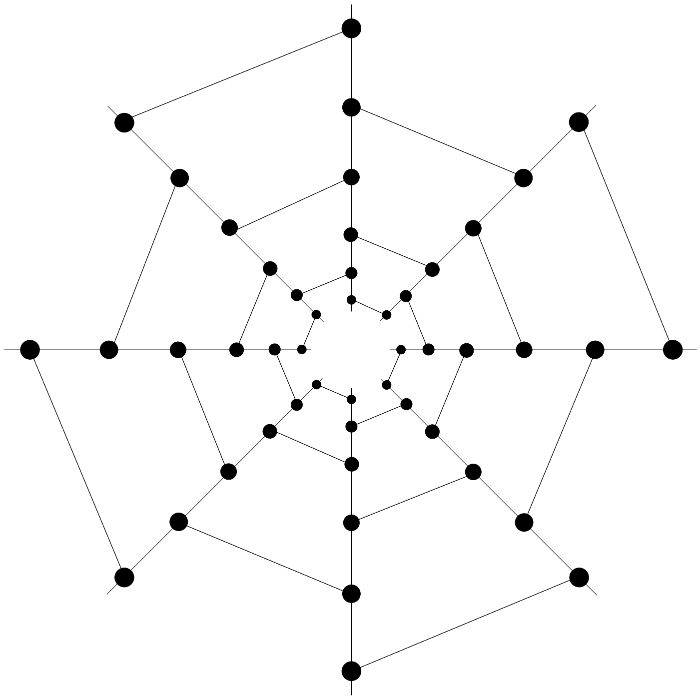


Figure 3.16: A plane embedding in a center-cylindric perspective of the uniquely determined planar strip of the vertex type (6^3) with the parameters $\kappa_\infty = 8$ and $p = 8$.

It remains the case that u and v are adjacent whereas u' and v' are not adjacent. Since Σ is v-transitive, there is an automorphism ϕ mapping u to u'. ϕ maps v onto a vertex $\neq u', v'$ of f' which must be adjacent to the vertex of f that is not adjacent to u (which is $\phi(v')$). It follows that the vertices $\neq u, v, u', v', \phi(u), \phi(v')$ in $V(f) \cup V(f')$ separate Σ, a contradiction to the fact that Σ is 3-connected.

Altogether, it has been shown that, whenever there are two 4-faces $f \neq f'$ of Σ, there is at most one edge connecting them, and that no 8,8-edge is adjacent with two vertices of the same 4-face. It follows that there is a natural bijection between the edges of Σ_I and the 8,8-edges of Σ and that Σ_I has the vertex type (4^4). Furthermore, since the 8,8-edges of Σ form an edge-orbit of Σ, Σ_I is e-transitive. From Theorem 3.18 it follows that if $\kappa_\infty(\Sigma) \geq 3$, then Σ_I is the uniquely determined planar graph of the vertex type (4^4) with $\lambda_\infty = 2\kappa_\infty$. If $\kappa(\Sigma_I) \leq 2$, then Σ_I must be the crossed strip.

The structure of Σ is uniquely determined by the structure of Σ_I. This is due to the fact that consecutive edges (in the cyclic ordering of some vertex f of Σ_I) belong to 8,8-edges of Σ that are incident with some adjacent vertices of the face f of Σ. Otherwise, one gets a contradiction with Proposition 3.6 (it is easy to find disjoint paths in Σ_I joining the endvertices other than f of two exclusive pairs of consecutive edges (in the cyclic ordering of the edges around f in Σ_I) and some disjoint paths in Σ that are mapped onto the paths in Σ_I). m is even, since $m = 2\kappa_\infty(\Sigma_I)$, which can easily be proved with the help of Lemma 3.12.

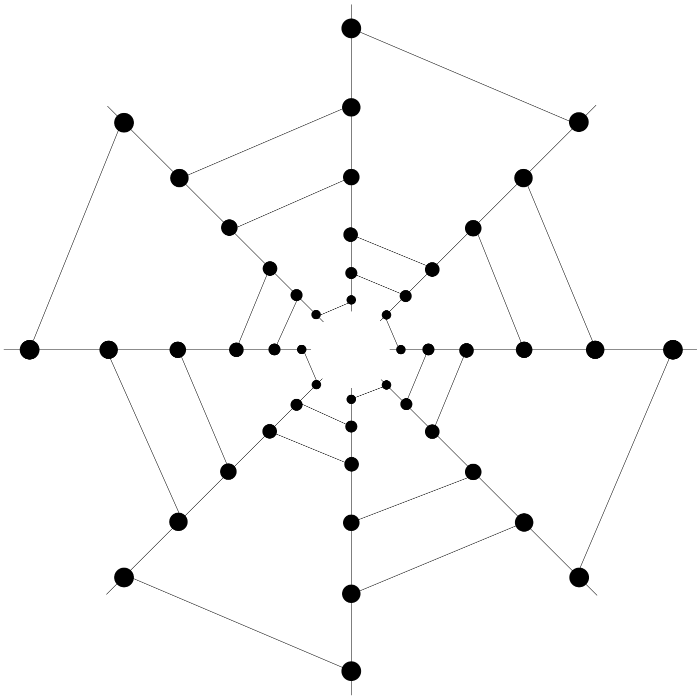


Figure 3.17: A plane embedding in a center-cylindric perspective of the uniquely determined planar strip of the vertex type (4.8^2) with the parameter $\kappa_\infty = 8$.

It follows by an index orgy that gives a formal description of Fig. 3.18 (similar as in the (3.4.6.4)-case of Theorem 3.13, see Fig. 3.2) that Σ has one of the structures described in the theorem. It is easy to see that Σ is v-transitive.

Now, if Σ has one of the structures given in the theorem, then it is immediate that Σ is a planar strip of the vertex type (4.8^2) with $m \geq 3$. □

3.3 Final Remarks

The above theorems give also a complete classification of all v-transitive, planar strips, since it has been shown that a v-transitive, planar strip Σ with $\kappa_\infty(\Sigma)$ has one of the vertex types covered by the above theorems. The theorem on the next page gives an overview about the results of this chapter with respect to this aspect.

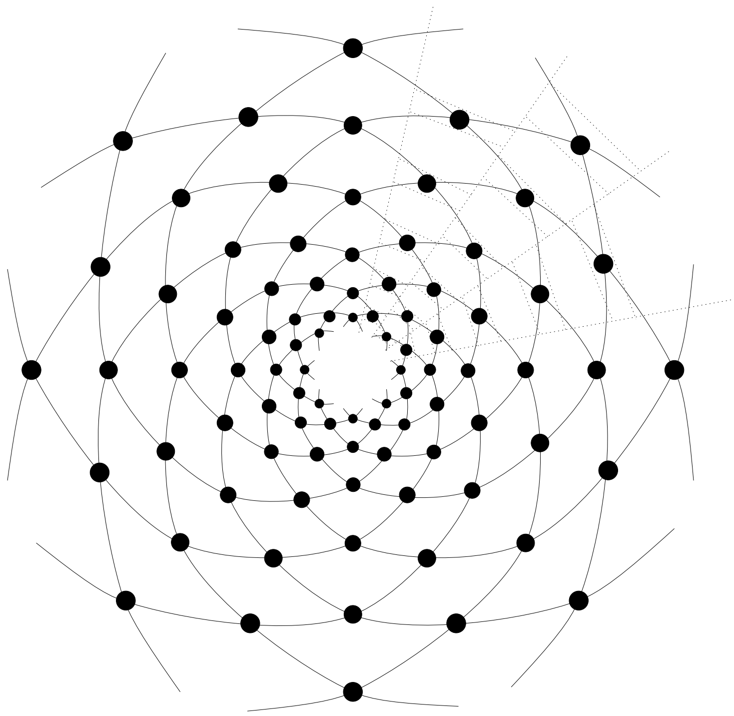


Figure 3.18: The bold graph represents Σ_I with $\kappa_\infty(\Sigma_I) = 8$. The overlying dotted graph shows how Σ can be derived from Σ_I.

Theorem 3.21.
Σ is a v-transitive, planar strip if and only if either

(1) $\kappa_\infty(\Sigma) = 1$ *and Σ is a 2-ray or*

(2) $\kappa_\infty(\Sigma) = 2$ *and Σ is one of the graphs shown in Fig. 3.1 or*

(3) $\kappa_\infty(\Sigma) \geq 3$ *and Σ either*

  (a) has the vertex type (3^6) *and is one of the graphs described in Theorem 3.14 or*
  (b) has the vertex type $(3^3.4^2)$ *and is one of the graphs described in Theorem 3.16 or*
  (c) has the vertex type $(3^2.4.3.4)$ *and is one of the graphs with the parameter* $p = 0$ *described in Theorem 3.17 or*
  (d) has the vertex type (4^4) *and is one of the graphs described in Theorem 3.18 or*
  (e) has the vertex type (3^6) *and is one of the graphs described in Theorem 3.19 or*
  (f) has the vertex type (4.8^2) *and is one of the graphs described in Theorem 3.20.*

□

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